New Zealand Mathematical Olympiad Committee

Sample Algebra Problems

by Ross Atkins

1. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence such that
$$a_{n+1} = a_n - a_{n-1}.$$
Given $a_1 = 2$, determine all possible values of a_{2017}.

2. For any x, y and z, show that
$$x^2 + y^2 + z^2 \geq xy + yz + zx.$$

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(3x + f(0)) = 3x^2$$
for all real x.

4. Suppose $p(x)$ is a polynomial of degree n, such that for $k = 0, 1, 2, 3, \ldots, n$ we have
$$p(k) = \frac{k}{k+1}.$$
Determine the value of $p(n+1)$. *(express your answer in terms of n)*

YOUR NAME

Q1S1

claim: the sequence $a_1, a_2, a_3, \dots$ is periodic with period $= 6$.

$\iff a_n = a_{n+6} \ \forall n.$

proof: since (a_{n+1}) depends only on the previous two terms, it suffices to show $a_7 = a_1$ and $a_8 = a_2$.

Let $a_2 = \lambda$

$\Rightarrow a_3 = a_2 - a_1 = \lambda - 2$

$\Rightarrow a_4 = a_3 - a_2 = (\lambda-2) - \lambda = -2$

$\Rightarrow a_5 = a_4 - a_3 = (-2) - (\lambda - 2) = -\lambda$

$\Rightarrow a_6 = a_5 - a_4 = (-\lambda) - (-2) = 2 - \lambda$

$\Rightarrow a_7 = a_6 - a_5 = (2-\lambda) - (-\lambda) = 2 = a_1$

$\Rightarrow a_8 = a_7 - a_6 = 2 - (2-\lambda) = \lambda = a_2$

$\square$

Now 2016 is a multiple of 6 so

$a_n = a_{n+2017} \Rightarrow a_{2017} = a_1 = 2$

QED

YOUR
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QIRI

sequence: $a_1, a_2, a_3, a_4, \ldots$

$$\boxed{a_{n+1} = a_n - a_{n-1}}$$

$a_n = a_{n-1} + a_{n+1}$

symetric ?

$a_1 = 2$

constant sequence?

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try $a_2 = 2$

$\Rightarrow a_3 = a_2 - a_1$
$= 2 - 2$
$= 0.$

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try $a_2 = 0$

$\Rightarrow a_3 = a_2 - a_1$
$= 0 - 2$
$= -2$

$\Rightarrow a_4 = a_3 - a_2$
$= (-2) - 0$
$= -2$

$\Rightarrow a_5 = a_4 - a_3$
$= (-2) - (-2)$
$= 0$

... hmm

YOUR NAME

QIR2

try $a_2 = 1$

$\Rightarrow a_3 = a_2 - a_1$
$= 1 - 2$
$= -1$

$\Rightarrow a_4 = a_3 - a_2$
$= (-1) - 1$
$= -2$

$\Rightarrow a_5 = a_4 - a_3$
$= (-2) - (-1)$
$= -1$

$\Rightarrow a_6 = (-1) - (-2)$
$= 1$

$\Rightarrow a_7 = 1 - (-1)$
$= 2$

$\Rightarrow a_8 = 2 - 1$
$= 1$

$\Rightarrow a_9 = 1 - 2$
$= -1$

... periodic?

n	a_n
1	2
2	1
3	-1
4	-2
5	-1
6	1
7	2
8	1
9	-1

but what if $a_2 \neq 1$?

...

YOUR NAME

Q1R3

$$a_1 = 2$$

let $a_2 = \lambda$

$\Rightarrow a_3 = \lambda - 2$

$\Rightarrow a_4 = (\lambda-2) - \lambda$

$= -2$

$\Rightarrow a_5 = (-2) - (\lambda-2)$

$= -\lambda$

$\Rightarrow a_6 = (-\lambda) - (-2)$

$= 2-\lambda$

$\Rightarrow a_7 = (2-\lambda) - (-\lambda)$

$= 2$

independent of λ

$\Rightarrow a_8 = 2 - (2-\lambda)$

$= \lambda$

yep! totally periodic.

Time to write this up neatly now

YOUR NAME

Q251

$$(x-y)^2 \geq 0$$

$$\Rightarrow x^2 - 2xy + y^2 \geq 0$$

$$\Rightarrow x^2 + y^2 \geq 2xy$$

Similarly $y^2 + z^2 \geq 2yz$

and $z^2 + x^2 \geq 2zx$

Add these together

$$\Rightarrow (x^2+y^2) + (y^2+z^2) + (z^2+x^2) \geq 2xy + 2yz + 2zx$$

$$\Rightarrow 2x^2 + 2y^2 + 2z^2 \geq 2xy + 2yz + 2zx$$

$$\Rightarrow x^2 + y^2 + z^2 \geq xy + yz + zx$$

QED.

YOUR NAME

Q2R1

RTP: $x^2+y^2+z^2 \geqslant xy+yz+zx$

try $z=0$

$\longrightarrow x^2+y^2+0 \geqslant xy+0+0$

true because $xy \leq \max(x,y)^2 \leq x^2+y^2$

too easy

WLOG $x \geqslant y \geqslant z$

try: $y=z$

$\longrightarrow x^2+y^2+y^2 \geqslant xy+y^2+xy$

$\iff x^2+y^2 \geqslant 2xy$

$\iff x^2-2xy+y^2 \geqslant 0$

$\iff \underline{\underline{(x-y)^2 \geqslant 0}}$!! great

Similarly $x^2+z^2 \geqslant 2xz$

and $y^2+z^2 \geqslant 2yz$

yeah - this'll work...

YOUR NAME

Q351

$$f:\mathbb{R}\to\mathbb{R}$$

$$f(3x+f(0)) = 3x^2 \quad \forall x\in\mathbb{R}$$

sub: $x = \frac{-f(0)}{3} \Rightarrow f\left(3\left(\frac{-f(0)}{3}\right)+f(0)\right) = 3\left(\frac{-f(0)}{3}\right)^2$

$$\Rightarrow f(-f(0)+f(0)) = \frac{1}{3}f(0)^2$$

$$\Rightarrow f(0) = \frac{1}{3}f(0)^2$$

$$\Rightarrow (3-f(0))f(0) = 0$$

$$\Rightarrow f(0) = 0 \text{ or } 3$$

Case 1: $f(0)=0$

sub: $x = \frac{a}{3}$

$\Rightarrow f(3(\frac{a}{3})+0) = 3(\frac{a}{3})^2$

$$\underline{\underline{f(a) = \frac{a^2}{3} \quad \forall a}}$$

Case 2: $f(0)=3$

sub: $x = \frac{a}{3}-1$

$f(3(\frac{a}{3}-1)+f(0)) = 3(\frac{a}{3}-1)^2$

$f(a-3+3) = 3(\frac{a^2}{9}-\frac{2a}{3}+1)$

$$\underline{\underline{f(a) = \frac{a^2}{3} - 2a + 3}}$$

Thus we have two candidate solutions:

$$f(x) = \frac{x^2}{3} \quad \& \quad f(x) = \frac{x^2}{3} - 2x + 3$$

Checking $f(x) = \frac{x^2}{3}$:

$$f(3x+f(0)) = f\left(3x+\frac{0^2}{3}\right)$$

$$= f(3x) = \frac{(3x)^2}{3}$$

$$= 3x^2 \checkmark$$

as required.

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Q352

Checking $f(x) = \frac{x^2}{3} - 2x + 3$

$$f(3x + f(0)) = f\left(3x + \left(\frac{0^2}{3} - 2\times 0 + 3\right)\right)$$

$$= f(3x + 3)$$

$$= \frac{(3x+3)^2}{3} - 2(3x+3) + 3$$

$$= \frac{9x^2 + 18x + 9}{3} - 6x - 6 + 3$$

$$= 3x^2 + 6x + 3 - 6x - 3$$

$$= \underline{\underline{3x^2}} \checkmark$$ as required.

Final Answer: 2 such functions

① $f(x) = \frac{x^2}{3} \quad \forall x \in \mathbb{R}$

and ② $f(x) = \frac{x^2}{3} - 2x + 3 \quad \forall x \in \mathbb{R}$

YOUR NAME

Q3R1

$$f(3x+f(0)) = 3x^2$$

$x=0 \Rightarrow f(f(0)) = 0$

$x=1 \Rightarrow f(3+f(0)) = 3$

$x=2 \Rightarrow f(6+f(0)) = 12$

$x=3 \Rightarrow f(9+f(0)) = 27$

$x=-1 \Rightarrow f(-3+f(0)) = 3 = f(3+f(0))$

generally: $\boxed{f(\lambda+f(0)) = f(-\lambda+f(0)) \quad \forall \lambda}$

because: $x = \frac{\lambda}{3} \Rightarrow f(\lambda+f(0)) = \frac{\lambda^2}{3}$

and $x = \frac{-\lambda}{3} \Rightarrow f(-\lambda+f(0)) = \frac{\lambda^2}{3}$

try $x = \frac{-f(0)}{3}$

$\Rightarrow f\left(3\left(\frac{-f(0)}{3}\right)+f(0)\right) = 3\left(\frac{-f(0)}{3}\right)^2$

$\Rightarrow f(0) = \frac{1}{3} f(0)^2$

$\Rightarrow 3f(0) - f(0)^2 = 0$

$\Rightarrow (3-f(0))f(0) = 0$

$\Rightarrow \underline{f(0) = 3 \text{ or } 0.}$

Case 1: $f(0) = 3$

$\Rightarrow f(3x+3) = 3x^2$

$x = \frac{\lambda-3}{3} \Rightarrow f(\lambda) = 3\left(\frac{\lambda-3}{3}\right)^2$

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Q3R2

$$f(\lambda) = 3\left(\frac{\lambda-3}{3}\right)^2$$
$$= \frac{1}{3}(\lambda^2 - 6\lambda + 9)$$
$$= \frac{\lambda^2}{3} - 2\lambda + 3.$$

Case 2: $f(0) = 0$

$$\Rightarrow f(3x + 0) = 3x^2$$

$$x = \frac{\lambda}{3} \Rightarrow f(\lambda) = \frac{\lambda^2}{3}$$

great.

YOUR
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Q4S1

P poly of degree n.

$$\boxed{P(k) = \frac{k}{k+1} \quad \forall\, k = 0, 1, 2, \ldots, n.}$$

Consider $Q(x) = (x+1)P(x) - x$.

$Q(x)$ is a poly of degree $n+1$.

Also note that for any $k = 0, 1, 2, \ldots, n$

we have

$$\begin{aligned} Q(k) &= (k+1)P(k) - k \\ &= (k+1)\frac{k}{k+1} - k \\ &= k - k \\ &= \underline{\underline{0}}. \end{aligned}$$

Therefor k is a root of $Q(x)$ $\forall\, k = 0, 1, \ldots, n$.

By FTOA this means

$$\boxed{Q(x) = a x (x-1)(x-2)(x-3)\ldots(x-n)}$$

To determine $a \in \mathbb{R}$, consider

$$Q(-1) = (-1+1)P(-1) + (-1) = +1$$

$$\therefore\ +1 = a(-1)(-1-1)(-1-2)(-1-3)\ldots(-1-n)$$

$$= a(-1)^{n+1}(n+1)!$$

$$\therefore\ a = \frac{(-1)^{n+1}}{(n+1)!}$$

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Q4 S2

$$\boxed{Q(x) = \frac{(-1)^{n+1}}{(n+1)!} x(x-1)(x-2)\dots(x-n)}$$

$$\therefore Q(n+1) = \frac{(-1)^{n+1}}{(n+1)!}(n+1)n(n-1)\dots 1$$

$$= \frac{(-1)^{n+1}}{(n+1)!}(n+1)!$$

$$= \underline{\underline{(-1)^{n+1}}}$$

But $Q(n+1) = ((n+1)+1)P(n+1) - (n+1)$

$$\Rightarrow (-1)^{n+1} = (n+2)P(n+1) - (n+1)$$

$$\Rightarrow P(n+1) = \frac{n+1+(-1)^{n+1}}{n+2}$$

ie. $P(n+1) = \frac{n+2}{n+2} = 1$ when n is odd

and. $P(n+1) = \frac{n}{n+2}$ when n is even.

YOUR NAME

Q4.R1

$\deg(p) = n$

try $n=0 \Rightarrow p$ is constant

$p(k) = \frac{k}{k+1}$ for $k=0$.

$\Rightarrow p(0) = 0$

$\Rightarrow p(x) = 0$ the zero poly only.

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try $n=1 \Rightarrow p(x) = ax+b$

$p(k) = \frac{k}{k+1}$ for $k=0,1$

$\Rightarrow p(0) = 0$ $\therefore b=0$

$p(1) = \frac{1}{2}$ $\therefore a+b = \frac{1}{2}$

$\Rightarrow p(x) = \frac{1}{2}x + 0$

$p(n+1) = p(2) = \frac{1}{2}(2) = 1.$

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try $n=2 \Rightarrow p(x) = ax^2+bx+c$

$p(k) = \frac{k}{k+1}$ for $k=0,1,2$

ie. $p(0)=0$, $p(1)=\frac{1}{2}$, $p(2)=\frac{2}{3}$.

$\Rightarrow c=0$, $a+b+c=\frac{1}{2}$, $4a+2b+c=\frac{2}{3}$.

YOUR NAME

Q4R2

$c = 0$

$a + b = \frac{1}{2}$

$4a + 2b = \frac{2}{3}$

$2a = \frac{2}{3} - 1 = \frac{-1}{3}$

$a = -\frac{1}{6}$

$b = \frac{2}{3}$

$\Rightarrow P(x) = -\frac{1}{6}x^2 + \frac{2}{3}x + 0$

$P(3) = -\frac{1}{6}(9) + \frac{2}{3}(3) + 0$

$= -\frac{3}{2} + 2$

$= \frac{1}{2}$

$P(k) = \dfrac{k}{k+1}$

$\Rightarrow (k+1)P(k) = k$ [for $k = 0, 1, 2, \dots, n$]

$(k+1)P(k) - k = 0$

Consider $[(x+1)P(x) - x]$ has $n+1$ distinct roots

and degree $= n+1$

Apply FTOA! I get it...