New Zealand Mathematical Olympiad Committee

NZMO Round Two 2025 — Solutions

1. **Problem:** Find all pairs of positive integers m and n such that the centres of the unit squares in a m by n grid of unit squares can be paired up so that the distance between the centres of each pair is exactly 2.

   (A unit square has side length 1.)

   Solution: (Tony Wang)
   We will show that the answer is all pairs (a, b) where either a or b (or both) is a multiple of 4. First, partition the grid into four subgrids A, B, C, and D

A	B	A	B	A	B	
C	D	C	D	C	D	
A	B	A	B	A	B	
C	D	C	D	C	D	

   Note that, for any given square s, all the square centres that are exactly 2 away from the centre of s are in the same partition as s itself. This means that s must be paired with some square which is in the same partition as itself. Hence, each partition of squares must have an even number of squares. Meanwhile, if each partition of squares has an even number of squares, one of the dimensions of the subgrid must be even, and hence we can pair up the squares along that even dimension.

   Hence, it suffices to find all values of a and b which create four subgrids which all have an even number of squares. Consider a modulo 4: if $a \equiv 0 \pmod 4$, then all subgrids will have an even dimension along the axis of a. Otherwise, at least one of the subgrids will have an odd dimension along the axis of a. The same reasoning holds for b. Hence, if neither a nor b are multiples of 4, then one of the partitions will have an odd number of squares. However, if either a or b is a multiple of 4, then all partitions will have an even number of squares. Hence, we have shown that the answer is all pairs (a, b) where either a or b (or both) is a multiple of 4.

2. **Problem:** For which positive integers n, does there exist a sequence of real numbers $(x_1, x_2, \ldots, x_n)$ such that

   - $-2 < x_i < 2$ for all i,
   - $x_1 + x_2 + x_3 + \cdots + x_n = 0$, and
   - $x_1^4 + x_2^4 + x_3^4 + \cdots + x_n^4 \geqslant 32$.

   Solution: (Eric Liang)
   Note that if $n = j$ works then $n > j$ also works for a positive integer j as we can just set $x_i = 0$ for $n \geq i > j$ and have $x_1, \ldots, x_n$ be the sequence that worked for n.

   Consider $n = 4$. We take $x_1, x_2 = \sqrt[4]{8}$ and $x_3, x_4 = -\sqrt[4]{8}$ and all conditions are satisfied. Thus all $n \geq 4$ works.

Now if $n = 2$, we note $|x_1|, |x_2| < 2$ and thus $x_1^4, x_2^4 < 16$ so $x_1^4 + x_2^4 < 32$. Contradiction.

If $n = 3$, then wlog $x_1, x_2 \geq 0$ and $x_3 \leq 0$ (as flipping the signs won't affect any of the conditions).

Now as $x_1 + x_2 = -x_3 < 2$, we get that $(x_1 + x_2)^4 < 16$. But if we expand this out we get $16 > (x_1+x_2)^4 = x_1^4 + 4x_1^3x_2 + 6x_1^2x_2^2 + 4x_1x_2^3 + x_2^4 \geq x_1^4 + x_2^4$ (as $x_1, x_2 \geq 0$).

Thus $x_1^4 + x_2^4 < 16$ but also note that $x_3^4 < 16$, thus $x_1^4 + x_2^4 + x_3^4 < 32$. Contradiction.

Thus, a sequence only exists for integers $n \geq 4$.

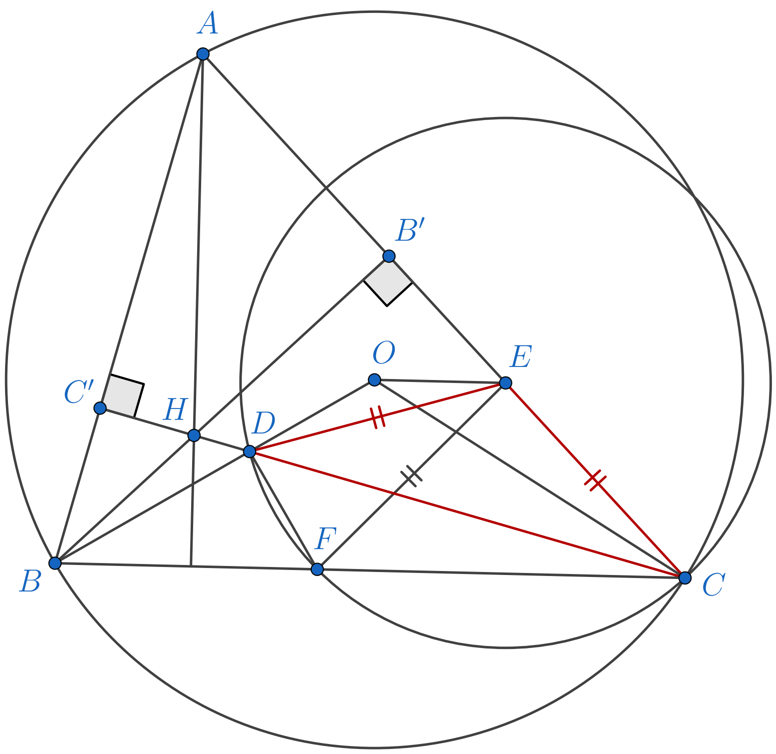
3. **Problem:**

Let ABC be an acute scalene triangle with $AC > BC > AB$. Let the orthocentre be H and circumcentre be O. Suppose that lines BO and CH intersect at a point D. Point E (where $E \neq C$) lies on side AC so that $OECD$ is cyclic. Point F (where $F \neq C$) lies on side BC such that $CE = FE$. Prove that $BHDF$ is cyclic.

(The *orthocentre* of a triangle is the point of intersection of its altitudes.)

Solution: (Nico McKinlay & George Zhu)

Let $\alpha = \angle BAC$. Let BB' and CC' be altitudes in triangle ABC, as shown.



Claim. Triangle CDE is isosceles with $CE = DE$.

Proof.

$$\begin{aligned} \angle DEC &= \angle DOC && (OECD \text{ cyclic}) \\ &= \angle BOC \\ &= 2\angle BAC && (\text{angle at centre/circumference}) \\ &= 2\alpha \end{aligned}$$

$$\begin{aligned} \angle DCE = \angle C'CA &= 90^\circ - \angle C'AC && (\text{angles in } \triangle C'CA) \\ &= 90^\circ - \alpha \end{aligned}$$

$$
\begin{aligned}
\angle CDE &= 180^\circ - \angle DEC - \angle DCE && \text{(angle sum in } \triangle CDE\text{)} \\
&= 180^\circ - 2\alpha - (90^\circ - \alpha) \\
&= 90^\circ - \alpha \\
&= \angle DCE. && \square
\end{aligned}
$$

Since $CE = DE$ and $CE = FE$, we have $CE = DE = FE$, therefore E is the circumcentre of triangle CDF. Consequently,

$$
\begin{aligned}
\angle HDF &= 180^\circ - \angle CDF && \text{(angles on a line)} \\
&= 180^\circ - \tfrac{1}{2} \cdot \angle CEF && \text{(angle at centre/circumference)} \\
&= 180^\circ - \tfrac{1}{2} \cdot (180^\circ - 2\angle ECF) && \text{(angle sum in isosceles } \triangle CEF\text{)} \\
&= 180^\circ - \tfrac{1}{2} \cdot (180^\circ - 2\angle B'CB) \\
&= 180^\circ - (90^\circ - \angle B'CB) \\
&= 180^\circ - \angle B'BC && \text{(angles in } \triangle B'BC\text{)} \\
&= 180^\circ - \angle HBF
\end{aligned}
$$

so $BHDF$ is cyclic.

4. **Problem:** The function $r_n(x)$ is the remainder when x is divided by n, where $0 \le r_n(x) < n$. For which n does there exists some ordering $\{a_1, \ldots, a_{n-1}\}$ of $\{1, 2, \ldots, n-1\}$ such that $\{r_n(a_1), r_n(2 \times a_2), , \ldots, r_n((n-1) \times a_{n-1})\}$ is an ordering of $\{1, 2, \ldots, n-1\}$?

   (An ordering of $\{1, 2, \ldots, n-1\}$ is the sequence of numbers 1 to $n-1$ in some order.)

   Solution: (James Xu)

   Notice that $r_n(x)$ is just x modulo n. Therefore

$$\prod_i r_n(ia_i) \equiv \prod_i ia_i \pmod n$$

   For primes p, apply Wilson's theorem to see that we must have

$$\prod_i ia_i \equiv -1 \pmod p$$

   However,

$$\prod_i ia_i = \prod_i i \prod_i a_i \equiv -1^2 \equiv 1 \pmod p$$

   So the only possibility in that case is $p = 2$.

   Now, if n is composite, then let n be the minimal solution. Let $n = pq$ for prime p. Notice that we must have $(p-1)q$ numbers in $\{a_1, 2a_2, 3a_3, \ldots, (n-1)a_{n-1}\}$ not divisible by p. The $q-1$ numbers of the form $(kp)a_{kp}$ are divisible by p and there are only $q-1$ multiples of p in $\{1, 2, \ldots, pq-1\}$. Therefore they must be the only numbers divisible by p, so $\{a_{kp} | 1 \le k < q\} = \{kp | 1 \le k < q\}$. Now, $p \nmid q$ as otherwise all of $(kp)a_{kp}$ are multiples of p^2, which is not true.

   Let $c \equiv \frac{1}{p} \pmod q$. Consider $\{\frac{a_p}{p}, \frac{a_{2p}}{p}, \ldots, \frac{a_{(q-1)p}}{p}\} = \{1, 2, \ldots, q-1\}$.

Then,

$$\{\frac{a_p}{p}, \frac{2a_{2p}}{p}, \ldots, \frac{(q-1)a_{(q-1)p}}{p}\} = \{\frac{pa_p}{p^2}, \frac{2pa_{2p}}{p^2}, \ldots, \frac{(q-1)pa_{(q-1)p}}{p^2}\}$$

$$\equiv c\{1, 2, 3, \ldots, q-1\} \pmod q$$

As $(kp)a_{kp} \equiv k'p \pmod{pq}$ for some k', and $ka_{kp} \equiv k' \pmod q$

Since $gcd(c, q) = 1$

$$c\{1, 2, 3, \ldots, q-1\} \equiv \{1, 2, 3, \ldots, q-1\} \pmod q$$

in some order. This is a solution for $n = q$, which contradicts the minimality of the solution for $n = pq$. Therefore no such solution $n = pq$ can exist.

Hence, $n = 2$ is the only solution.

5. **Problem:** Let a, b, c be positive real numbers satisfying $abc = 1$. Determine the smallest possible value of

$$\frac{a^2 + 2025}{a^3(b+c)} + \frac{b^2 + 2025}{b^3(c+a)} + \frac{c^2 + 2025}{c^3(a+b)}$$

Solution: (Eric Liang)

Note,

$$\begin{aligned}
\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} &= \frac{(\frac{1}{a})^2}{a(b+c)} + \frac{(\frac{1}{b})^2}{b(a+c)} + \frac{(\frac{1}{c})^2}{c(a+b)} \\
&\geq \frac{(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}{2(ab + ac + bc)} && \text{(Cauchy-Schwarz)} \\
&= \frac{(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}{2(\frac{1}{c} + \frac{1}{b} + \frac{1}{a})} && (abc = 1) \\
&= \frac{(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})}{2} \\
&\geq \frac{3}{2\sqrt[3]{abc}} && \text{(AM-GM)} \\
&= \frac{3}{2}
\end{aligned}$$

Now, $(a + b + c)^2 \geq 3(ab + ac + bc)$ for positive reals a, b, c. (†)

So we also get,

$$\begin{aligned}
\frac{1}{a^2(b+c)} + \frac{1}{b^2(a+c)} + \frac{1}{c^2(a+b)} &\geq \frac{(\frac{1}{a})^2}{(b+c)} + \frac{(\frac{1}{b})^2}{(a+c)} + \frac{(\frac{1}{c})^2}{(a+b)} \\
&\geq \frac{(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}{2(a + b + c)} && \text{(Cauchy-Schwarz)} \\
&= \frac{(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}{2(\frac{1}{b}\cdot\frac{1}{c} + \frac{1}{a}\cdot\frac{1}{c} + \frac{1}{a}\cdot\frac{1}{b})} && (\dagger) \\
&\geq \frac{3}{2}
\end{aligned}$$

Finally, as $a^2 + 1 \geq 2a$, we get

$$\begin{aligned}\frac{a^2+2025}{a^3(b+c)} + \frac{b^2+2025}{b^3(a+c)} + \frac{c^2+2025}{c^3(a+b)} &\geq \frac{2a+2024}{a^3(b+c)} + \frac{2b+2024}{b^3(a+c)} + \frac{2c+2024}{c^3(a+b)} \\ &\geq 2 \cdot \frac{3}{2} + 2024 \cdot \frac{3}{2} \\ &= 3039\end{aligned}$$

Equality can be achieved when $a = b = c = 1$.