New Zealand Mathematical Olympiad Committee

# NZMO Round Two 2025 — Instructions

*Exam date: $5^{th}$ September*

## General Instructions

1. You have 3 hours to work on the exam.
2. There are 5 problems, each worth equal marks. You should attempt all 5 problems. You may work on them in any order.
3. Geometrical instruments (ruler and compasses) may be used. Calculators, phones, computers and electronic devices of any sort are not permitted.
4. Write your solutions on the paper provided. Let the supervisor know if you need more. Write your name at the top of every page as you start using it.
5. Full written solutions — not just answers — are required, with complete proofs of any assertions you make. Your marks will depend on the clarity of your mathematical presentation. Work in rough first, and then draft a neat final version of your best attempt.
6. Make sure you fill in your details on the Declaration Form. Hand in all of your rough work, in addition to your neat solutions. The declaration form is used as a cover sheet for your submission.
7. At the end of the exam, remain seated quietly until all scripts have been collected and the supervisor indicates that you are free to move.
8. You may not take the question paper from the exam room.
9. The contest problems are to be kept confidential until they are posted on the NZMOC webpage www.mathsolympiad.org.nz. Do not disclose or discuss the problems online until this has occurred. Typically this will be approximately one week after the exam has been sat.
10. **Do not turn over until told to do so**.

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- There are 5 problems. You should attempt to find solutions for as many as you can.
- Solutions (that is, answers with justifications) and not just answers are required for all problems, even if they are phrased in a way that only asks for an answer.
- Read and follow the "General instructions" accompanying these problems.

**Problems**

1. Find all pairs of positive integers $m$ and $n$ such that the centres of the unit squares in a $m$ by $n$ grid of unit squares can be paired up so that the distance between the centres of each pair is exactly 2.

   (A unit square has side length 1.)

2. For which positive integers $n$, does there exist a sequence of real numbers $(x_1, x_2, \ldots, x_n)$ such that

   - $-2 < x_i < 2$ for all $i$,
   - $x_1 + x_2 + x_3 + \cdots + x_n = 0$, and
   - $x_1^4 + x_2^4 + x_3^4 + \cdots + x_n^4 \geqslant 32$.

3. Let $ABC$ be an acute scalene triangle with $AC > BC > AB$. Let the orthocentre be $H$ and circumcentre be $O$. Suppose that lines $BO$ and $CH$ intersect at a point $D$. Point $E$ (where $E \neq C$) lies on side $AC$ so that $OECD$ is cyclic. Point $F$ (where $F \neq C$) lies on side $BC$ such that $CE = FE$. Prove that $BHDF$ is cyclic.

   (The *orthocentre* of a triangle is the point of intersection of its altitudes.)

4. The function $r_n(x)$ is the remainder when $x$ is divided by $n$, where $0 \leq r_n(x) < n$. For which $n$ does there exists some ordering $\{a_1, \ldots, a_{n-1}\}$ of $\{1, 2, \ldots, n-1\}$ such that $\{r_n(a_1), r_n(2 \times a_2), , \ldots, r_n((n-1) \times a_{n-1})\}$ is an ordering of $\{1, 2, \ldots, n-1\}$?

   (An ordering of $\{1, 2, \ldots, n-1\}$ is the sequence of numbers 1 to $n-1$ in some order.)

5. Let $a, b, c$ be positive real numbers satisfying $abc = 1$. Determine the smallest possible value of

$$\frac{a^2 + 2025}{a^3(b+c)} + \frac{b^2 + 2025}{b^3(c+a)} + \frac{c^2 + 2025}{c^3(a+b)}$$