New Zealand Mathematical Olympiad Committee

NZMO Round Two 2024 — Solutions

1. **Problem:** At each vertex of a regular 14-gon, lies a coin. Initially 7 coins are heads, and 7 coins are tails. Determine the minimum number t such that it's always possible to turn over at most t of the coins so that in the resulting 14-gon, no two adjacent coins are both heads and no two adjacent coins are both tails.

   Solution A: (Ross Atkins)
   Number the vertices from 1 to 14. We need to make sure that all the even-numbered vertices are tails and all the odd numbered vertices are heads or vice versa.

   Lets look at the 7 odd numbered vertices.
   Without loss of generality assume that there are initially more heads than tails on the odd numbered vertices. *i.e.* we assume that there are initially at least 4 heads on odd numbered vertices. This means there are at most 3 tails on odd numbered vertices and at most 3 heads on even numbered vertices. So we flip all the tails on odd numbered vertices and all the heads on even numbered vertices then the total number of flips required is at most 6.

   If there are exactly 4 heads (and 3 tails) on odd numbered vertices, then we would require at least 3 flips to make all the odd numbered vertices the same, and similarly at least 3 flips to make the off numbered vertices the same. So we cannot *guarantee* to be able to achieve our goal in fewer than 6 flips.

2. **Problem:** Consider the sequence $a_1, a_2, a_3, \ldots$ defined by $a_1 = 2024^{2024}$ and for each positive integer n,

$$a_{n+1} = \left| a_n - \sqrt{2} \right| .$$

Prove that there exists an integer k such that $a_{k+2} = a_k$.
Here $|x|$ denotes the absolute value of x.

Solution A: (Ross Atkins)
As long as $a_n \geqslant \sqrt{2}$ we have $a_{n+1} = a_n - \sqrt{2}$ and so the sequence begins with an arithmetic progression with common difference $d = -\sqrt{2}$. Let k be the first index such that $a_k < \sqrt{2}$. So $a_{k-1} \geqslant \sqrt{2}$. Therefore

$$a_k = a_{k-1} - \sqrt{2} \geqslant \sqrt{2} - \sqrt{2} = 0.$$

Thus we have $0 \leq a_k < \sqrt{2}$. From this we can calculate $a_k - \sqrt{2} < 0$ and so $\left| a_k - \sqrt{2} \right| = \sqrt{2} - a_k$.

$$\implies a_{k+1} = \left(\sqrt{2} - a_k \right)$$

Since $a_{k+1} = \left(\sqrt{2} - a_k \right) \leqslant \left(\sqrt{2} - 0 \right) = \sqrt{2}$, this means that $\left(a_{k+1} - \sqrt{2} \right)$ is negative and thus

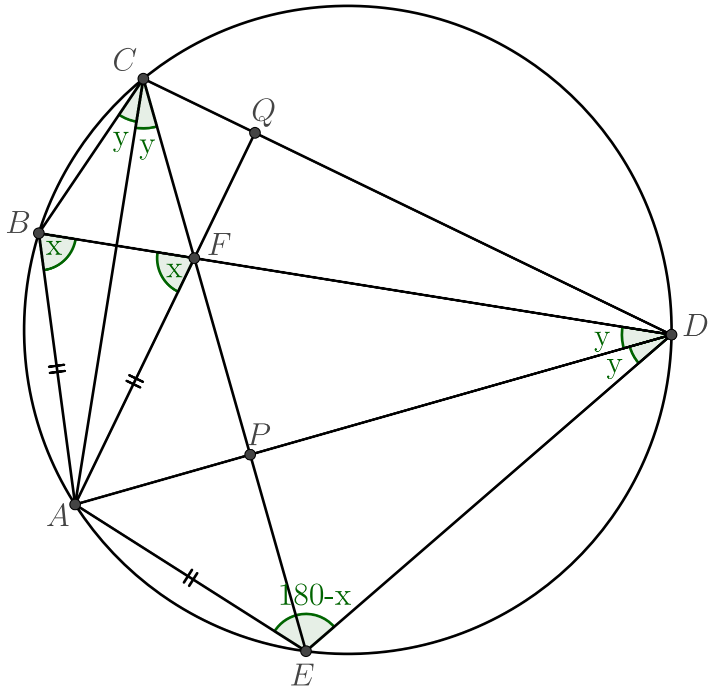
$$\begin{aligned} a_{k+2} &= \left| a_{k+1} - \sqrt{2} \right| \\ &= \left(\sqrt{2} - a_{k+1} \right) \\ &= \left(\sqrt{2} - \left(\sqrt{2} - a_k \right) \right) \\ &= a_k. \end{aligned}$$

as required.

3. **Problem:** Let A, B, C, D, E be five different points on the circumference of a circle in that (cyclic) order. Let F be the intersection of chords BD and CE. Show that if $AB = AE = AF$ then lines AF and CD are perpendicular.

Solution A: (Ross Atkins)
Let $x = \angle ABF$ and let $y = \angle BCA$. Since $\triangle ABF$ is isosceles, we get $\angle BFA = x$. Since $AB = AE$, it follows that arcs BA and AE are equal. Since equal arcs subtend equal angles, every angle subtended by either arc AB or AE must be equal to $\angle BCA = y$.

$$\implies \angle BCA = \angle BDA = \angle ACE = \angle ADE = y.$$



Since opposite angles in a cyclic quadrilateral ($ABDE$) are supplimentary, we get $\angle ABD + \angle DEA = 180°$. Therefore $\angle DEA = 180° - x$. Now consider triangles AED and AFD. We have

$$\angle ADE = y = \angle ADF \text{ and } \angle AED = 180° - x = \angle AFD$$

and side AD is shared. Therefore these triangles are congruent: $\triangle AED \equiv \triangle AFD$. Hence

$$\angle EAD = \angle DAF = x - y. \qquad \text{(angle sum in } \triangle AFD)$$

Now let P be the intersection of AD and EF. Also let Q be the intersection of AF and CD. Since AP is the angle bisector of isosceles triangle AEF, we have

$$\begin{aligned} & \angle APF = 90°. \\ \implies & \angle AFP = 90° + y - x && \text{(angle sum in } \triangle AFP) \\ & \angle CFQ = 90° + y - x && \text{(vertically opposite)} \end{aligned}$$

Finally we get $\angle ECD = \angle EAD = x - y$ by the Bow Tie Theorem ($ACDE$ cyclic). Therefore $\angle FCQ = x - y$. Now consider the sum of the angles in triangle FCQ to get

$$\angle CQF + (x - y) + (90° + y - x) = 180°$$

$$\implies \angle CQF = 90°$$

as required.

4. **Problem:** Determine all positive integers n less than 2024 such that for all positive integers x, the greatest common divisor of $9x + 1$ and $nx + 1$ is 1.

Solution A: (Ross Atkins)
Let $m = (n - 9)$ and for the sake of contradiction assume m has a prime factor p with $p \neq 3$. Express $p = 9q + r$ where $0 \leq r \leq 8$ (r is the remainder when p is divided by 9 and q may be zero). Since $\gcd(9, p) = 1$ we must have $r = 1, 2, 4, 5, 7$ or 8. For each of these possibilties we will use a different choice for x to get our contradiction.

- if $r = 1$ then choose $x = q$ so that $(9x + 1) = 9q + 1 = p$.
- if $r = 2$ then choose $x = 5q + 1$ so that $(9x + 1) = 45q + 10 = 5p$.
- if $r = 4$ then choose $x = 7q + 3$ so that $(9x + 1) = 63q + 28 = 7p$.
- if $r = 5$ then choose $x = 2q + 1$ so that $(9x + 1) = 18q + 10 = 2p$.
- if $r = 7$ then choose $x = 4q + 3$ so that $(9x + 1) = 36q + 28 = 4p$.
- if $r = 8$ then choose $x = 8q + 7$ so that $(9x + 1) = 72q + 64 = 8p$.

In any we can choose x so that $(9x + 1)$ is a multiple of p. For this particular value of x we have

$$(nx + 1) = mx + (9x + 1),$$

and so $(nx + 1)$ is a multiple of p too (recall $p|m$). Therefore p would be a common factor of $(9x + 1)$ and $(nx + 1)$. This is a contradiction so no such p can exist.
So $m = (n - 9)$ cannot have any prime factors other than 3. Hence $m = 3^k$ or $m = -3^k$ for some integer $k \geq 0$. *i.e.*

$$n = 9 + m = 9 + 3^k \text{ or } 9 - 3^k$$

The positive integers of this form, less than 2024 are: $6, 8, 10, 12, 18, 36, 90, 252, 738$.

To show all these work, we now assume $n = 9 \pm 3^k$ for some integer $k \geq 0$, and let $g = \gcd(9x + 1, nx + 1)$. Since $9x + 1$ is not a multiple of 3, we cannot have g being a multiple of 3. However

$$g \mid (nx + 1) - (9x + 1) = \pm 3^k.$$

The only divisors of 3^k which are not a multiple of 3 are 1 and -1. Therefore $g = 1$ whenever $n = 9 \pm 3^k$.

5. **Problem:** Determine the least real number L such that

$$\frac{1}{a}+\frac{a}{b}+\frac{b}{c}+\frac{c}{d}\leqslant L$$

for all quadruples (a,b,c,d) of integers satisfying $1<a<b<c<d$.

Solution: (Ross Atkins)
Answer: 3. To solve this problem, two parts are required. Part A shows that $L=3$ works. Part B shows that no $L'<3$ works.

- **Part A** We show that for all quadruples (a,b,c,d) (with $1<a<b<c<d$) we have

$$\frac{1}{a}+\frac{a}{b}+\frac{b}{c}+\frac{c}{d}\leqslant 3.$$

We do this by considering the two cases $b=a+1$ and $b\geq a+2$ seperately.
If $b\geq a+2$ then we can get

$$\begin{aligned}\frac{1}{a}+\frac{a}{b}+\frac{b}{c}+\frac{c}{d}&\leqslant\frac{1}{a}+\frac{a}{a+2}+\frac{b}{c}+\frac{c}{d}\\&<\frac{1}{a}+\frac{a}{a+2}+1+1\\&=\frac{2-a}{a(a+2)}+3\quad\leqslant\quad 3\end{aligned}$$

The numerator $(2-a)$ is non-positive because $1<a$ and a is an integer.
If $b=a+1$ then we can get

$$\begin{aligned}\frac{1}{a}+\frac{a}{b}+\frac{b}{c}+\frac{c}{d}&=\frac{1}{a}+\frac{a}{a+1}+\frac{a+1}{c}+\frac{c}{d}\\&<\frac{1}{a}+\frac{a}{a+1}+\frac{a+1}{c}+1\\&\leqslant\frac{1}{a}+\frac{a}{a+1}+\frac{a+1}{a+2}+1\\&=\frac{2-a^2}{a(a+1)(a+2)}+3\quad<\quad 3.\end{aligned}$$

The numerator $(2-a^2)$ is negative because $1<a$ and a is an integer.
In either case we get $\frac{1}{a}+\frac{a}{b}+\frac{b}{c}+\frac{c}{d}\leqslant 3$ for all quadruples (a,b,c,d).

- **Part B** For the sake of contradiction, suppose there existed some $L' < 3$ which worked. We will show that there exists a quadruple (a, b, c, d) such that

$$\frac{1}{a} + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} > L'.$$

We will do this explicitly by choosing a, b, c, d to be large consecutive integers. If $L' = 3 - \epsilon$ then we have $\epsilon > 0$, and so there exists some integer n such that $n > \frac{3}{\epsilon}$. This ensures $\frac{1}{n+1}$, $\frac{1}{n+2}$ and $\frac{1}{n+3}$ are each smaller than $\frac{\epsilon}{3}$. Now consider $(a, b, c, d) = (n, n+1, n+2, n+3)$.

$$\begin{aligned}\frac{1}{a} + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} &> \frac{a}{b} + \frac{b}{c} + \frac{c}{d} \\ &= \frac{n}{n+1} + \frac{n+1}{n+2} + \frac{n+2}{n+3} \\ &= 3 - \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\right) \\ &> 3 - \left(\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}\right) \\ &= L'\end{aligned}$$

Contradiction.