New Zealand Mathematical Olympiad Committee

NZMO Round Two 2023 — Solutions

1. **Problem:** For any positive integer n let $n! = 1 \times 2 \times 3 \times \cdots \times n$. Do there exist infinitely many triples (p, q, r), of positive integers with $p > q > r > 1$ such that the product

$$p! \cdot q! \cdot r!$$

is a perfect square?

Solution A: (James Xu)
Yes. Let t be an arbitrary positive integer and consider the following perfect square:

$$(t!)!^2 = (t!)! \cdot (t! - 1)! \cdot t!.$$

So if we consider $(p, q, r) = (t!, t! - 1, t)$ then $p!q!r! = (t!)!^2$ which is a perfect square. Since the choice of t is arbitrary, there must be infinitely many such triples.

Solution B: (Josie Smith)
Yes. Consider the substitution $(p, q, r) = \big(6t^2, 6t^2 - 1, 3\big)$.

$$p! \cdot q! \cdot r! = \big(6t^2\big)! \cdot \big(6t^2 - 1\big)! \cdot 3! = \big((6t^2 - 1)! \times 6t\big)^2$$

which is a perfect square. Since the choice of t is arbitrary, there must be infinitely many such triples.

Solution C: (Michael Albert)
This solution shows a stronger result. For any positive integer k, there exist infinitely many q such that $(q + 1)! \cdot q! \cdot k$ is a perfect square. This can be seen by setting $q + 1 = kx^2$ for any x. This of course implies the required result.

Comment: There are many more triples (p, q, r) that work. For example $(p, q, r) = (10, 7, 6)$. But this does not matter because the problem only asked us to determine whether or not there are infinitely many triples that work.

2. **Problem:** Let a, b and c be positive real numbers such that $a+b+c = abc$. Prove that at least one of a, b or c is greater than $\frac{17}{10}$.

Solution A: (Ross Atkins)
Wlog assume $a \geq b \geq c$. Therefore $a+b+c \geq 3c$. Now for a proof by contradiction, assume $a \leq \frac{17}{10}$ and $b \leq \frac{17}{10}$. Since $\left(\frac{17}{10}\right)^2 = \frac{289}{100} < 3$, it follows that $\frac{17}{10} < \sqrt{3}$ and therefore $ab \leq 3$. However this implies '

$$3c \leq a+b+c = abc < 3c$$

which is a contradiction.

Solution B: (Michael Albert)
From $abc = a+b+c$ we get

$$1 = \frac{a+b+c}{abc} = \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab}$$

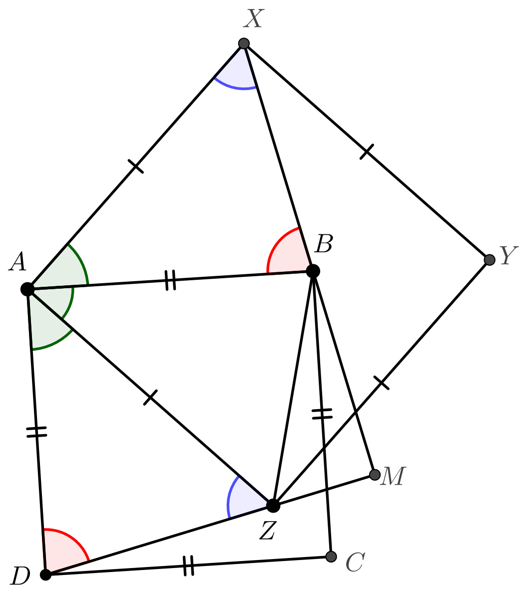
As all three terms are positive, at least one must be less than or equal to $1/3$ and, without loss of generality, we can assume $1/bc \leq 1/3$. But then $bc \geq 3$ and at least one of b or c must be greater than $\sqrt{3}$. Since $\sqrt{3} > 17/10$ we're done.

Solution C: (Ross Atkins)
By the AM-GM inequality we have $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ and thus $abc \geq 3\sqrt[3]{abc}$, which rearranges to give $\sqrt[3]{abc} \geq \sqrt{3}$. Now wlog $a \geq b \geq c$ and so

$$a = \sqrt[3]{a^3} \geq \sqrt[3]{abc} \geq \sqrt{3} > \frac{17}{10}.$$

3. **Problem:** Let $ABCD$ be a square (vertices labelled in clockwise order). Let Z be any point on diagonal AC between A and C such that $AZ > ZC$. Points X and Y exist such that $AXYZ$ is a square (vertices labelled in clockwise order) and point B lies inside $AXYZ$. Let M be the point of intersection of lines BX and DZ (extended if necessary). Prove that C, M and Y are collinear.

   Solution: (Kevin Shen)
   Since Z lies on diagonal AC, we have $\angle DAZ = 45°$ and $\angle ZAB = 45°$. Therefore B lies on diagonal AY of square $AXYZ$ and $\angle BAX = 45°$.



   Since $AB = AD$ and $AX = AZ$ and $\angle BAX = 45° = \angle DAZ$, we have congruent triangles

$$\triangle DAZ \equiv \triangle BAX \qquad (SAS).$$

   Therefore let $x = \angle ZDA = \angle XBA$ and $y = \angle AZD = \angle AXB$. By the angle sum in triangle DAZ we have $\angle DAZ + \angle AZD + \angle ZDA = 180°$. Therefore $x + y = 135°$. Now by the angle sum in quadrilateral $DAXM$ we get $\angle DAX + \angle AXM + \angle XMD + \angle MDA = 360°$. Therefore

$$\angle BMD = 90°.$$

   Hence $ABMCD$ is cyclic (the circle with diameter BD). Therefore

$$\angle DMC = \angle DAC = 45°.$$

   Also $AXYMZ$ is cyclic (the circle with diameter XZ). Therefore

$$\angle YMX = \angle YAX = 45°.$$

   Hence $\angle YMC = \angle YMX + \angle YMD + \angle DMC = 45° + 90° + 45° = 180°$ as required.

4. **Problem:** For any positive integer n, let $f(n)$ be the number of subsets of $\{1, 2, \ldots, n\}$ whose sum is equal to n. Does there exist infinitely many positive integers m such that $f(m) = f(m+1)$? (Note that each element in a subset must be distinct.)

   Solution: (Michael Albert)
   Let $S(n)$ be the set of such subsets. Consider the map from $S(n)$ to $S(n+1)$ that adds one to the largest element of each $A \in S(n)$. This map is an injection (needs proof but easy) and not a surjection provided that $S(n+1)$ contains a set whose largest and second largest elements differ by one. For even $n = 2k \geqslant 2$ this is true since we can take $\{k, k+1\} \in S(n+1)$ and for odd $n = 2k+1 \geqslant 5$ this is true since we can take $\{1, k, k+1\}$. So for $n \geqslant 5$, we must have $f(n) < f(n+1)$ and there do not exist infinitely many such pairs.

5. **Problem:** Let x, y and z be real numbers such that: $x^2 = y + 2$, and $y^2 = z + 2$, and $z^2 = x + 2$. Prove that $x + y + z$ is an integer.

Solution A: (Ross Atkins)
First we exclude -1 and 2:

- $x = 2$ implies $y = 2$ implies $z = 2$ implies $x = 2$.
- $x = -1$ implies $y = -1$ implies $z = -1$ implies $x = -1$.

In both these cases we have $x + y + z$ being an integer. So henceforth we assume none of x, y, z are 2 nor -1. Now let x, y, z be the roots of the following cubic equation.

$$(\lambda - x)(\lambda - y)(\lambda - z) = \lambda^3 - A\lambda^2 + B\lambda - C.$$

Applying Viete's formula to this cubic gives us $A = x + y + z$, $B = xy + yz + zx$ and $C = xyz$. This means that $x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + zx) = A^2 - 2B$. Now sum the three given equations ($x^2 = y + 2$ and $y^2 = z + 2$ and $z^2 = x + 2$) to get

$$A^2 - 2B = x^2 + y^2 + z^2 = (y + 2) + (z + 2) + (x + 2) = A + 6.$$

$$A^2 - A - 2B = 6. \tag{1}$$

Next rearrange the equations to be $x^2 - 1 = y + 1$ and $y^2 - 1 = z + 1$ and $z^2 - 1 = x + 1$. These can then be multiplied to get

$$\begin{aligned}(x^2 - 1)(y^2 - 1)(z^2 - 1) &= (y + 1)(z + 1)(x + 1)\\ (x - 1)(x + 1)(y - 1)(y + 1)(z - 1)(z + 1) &= (y + 1)(z + 1)(x + 1)\\ (x - 1)(y - 1)(z - 1) &= 1\\ xyz - (xy + yz + zx) + (x + y + z) - 1 &= 1\end{aligned}$$

$$B - A + 2 = C \tag{2}$$

In the above algebraic manipulation, we are allowed to cancel the $(x + 1)(y + 1)(z + 1)$ factor because none of x, y, z are equal to -1. Finally rearrange the equations to be $x^2 - 4 = y - 2$ and $y^2 - 4 = z - 2$ and $z^2 - 4 = x - 2$. These can then be multiplied to get

$$\begin{aligned}(x^2 - 4)(y^2 - 4)(z^2 - 4) &= (y - 2)(z - 2)(x - 2)\\ (x - 2)(x + 2)(y - 2)(y + 2)(z - 2)(z + 2) &= (y - 2)(z - 2)(x - 2)\\ (x + 2)(y + 2)(z + 2) &= 1\\ xyz + 2(xy + yz + zx) + 4(x + y + z) + 8 &= 1\\ C + 2B + 4A &= -7\end{aligned}$$

$$C = -4A - 2B - 7 \tag{3}$$

In the above algebraic manipulation, we are allowed to cancel the $(x-2)(y-2)(z-2)$ factor because none of x, y, z are equal to 2. Combining equations (2) and (3) gives us $B-A+2 = C = -4A-2B-7$ which rearranges to give us $B = -A - 3$. Substituting this into 1 gives us.

$$\begin{aligned}A^2 - A - 2B &= 6\\ A^2 - A - 2(-A - 3) &= 6\\ A(A + 1) &= 0.\end{aligned}$$

Therefore $A = 0$ or $A = -1$ both of which are integers. Since $A = x + y + z$ we are done.

Solution B: (Ross Atkins)
Consider the polynoimal P defined by

$$\begin{aligned} P(\lambda) &= \lambda^8 - 8\lambda^6 + 20\lambda^4 - 16\lambda^2 - \lambda + 2 \\ &= (\lambda+1)(\lambda-2)(\lambda^3 - 3\lambda + 1)(\lambda^3 + \lambda^2 - 2\lambda - 1). \end{aligned}$$

If we substitute $z = y^2 - 2$ into $z^2 = x + 2$ gives us $(y^2 - 2)^2 = x + 2$. Then substitute $y = x^2 - 2$ to get $\left((x^2-2)^2 - 2\right)^2 = x + 2$. Expanding gives

$$x^8 - 8x^6 + 20x^4 - 16x^2 - x + 2 = 0.$$

Therefore x is a root of the polynomial P. By symmetry we must have all of x, y, z being roots of P. Now we consider cases:

- Case 1: at least one of x, y, z is equal to -1. Wlog assume $x = -1$. Using $y = x^2 - 2$ we get $y = -1$. Then using $z = y^2 - 2$ we get $z = -1$. In this case we get $(x, y, z) = (-1, -1, -1)$ which has sum -3 which is an integer.
- Case 2: at least one of x, y, z is equal to 2. Wlog assume $x = 2$. Using $y = x^2 - 2$ we get $y = 2$. Then using $z = y^2 - 2$ we get $z = 2$. In this case we get $(x, y, z) = (2, 2, 2)$ which has sum 6 which is an integer.
- Case 3: at least one of x, y, z is a root of $(\lambda^3 - 3\lambda + 1)$. Wlog assume $x^3 - 3x + 1 = 0$. Note that $z = y^2 - 2 = (x^2 - 2)^2 - 2 = x^4 - 4x + 2$. Now consider the sum of x and $y = x^2 - 2$ and $z = x^4 - 4x + 2$,

  $$x + y + z = x + (x^2 - 2) + (x^4 - 4x^2 + 2) = x^4 - 3x^2 + x = (x^3 - 3x + 1)x.$$

  But since $x^3 - 3x + 1 = 0$ this means $x + y + z = 0$ in this case.
- Case 4: some two of x, y, z are the same. Wlog assume $x = y$ therefore $x = y = x^2 - 2$. Hence

  $$0 = x^2 - x - 2 = (x - 2)(x + 1).$$

  and so $x = -1$ or $x = 2$, and this was covered in cases 1 and 2.
- Case 5: Since x, y, z are all roots of P, the only remaining possibility is that x, y and z are distinct roots of $\lambda^3 + \lambda^2 - 2\lambda - 1$. By Viete's formula this means that the sum of the roots is -1 in this case.

In all cases we conclude that $x + y + z$ is an integer.

Comment: Each of the eight roots of P lead to genuine solutions. For example the roots of $\lambda^3 - 3\lambda + 1$ are approximately $0.3473\ldots$, $-1.8794\ldots$ and $1.5321\ldots$, and so

$$(x, y, z) = (0.3473\ldots, -1.8794\ldots, 1.5321\ldots)$$

is a valid solution. Similiarly, the roots of $\lambda^3 + \lambda^2 - 2\lambda - 1$ are approximately $1.2470\ldots$, $-0.44504\ldots$ and $-1.8019\ldots$, and so

$$(x, y, z) = (1.2470\ldots, -0.44504\ldots, -1.8019\ldots)$$

is also a valid solution.