New Zealand Mathematical Olympiad Committee

NZMO Round Two 2022 — Solutions

1. **Problem:** Find all integers a, b such that

$$a^2 + b = b^{2022}.$$

Solution: (Ethan Ng)
Let $g = \gcd(b, b^{2021} - 1)$. Since we cannot have both b and $(b^{2021} - 1)$ being zero, g must be a positive integer. Since $g|b$ we must have $g|b^{2021}$ and therefore

$$g \mid (b^{20221}) - (b^{2021} - 1) = 1.$$

Hence $g = 1$ and thus b and $(b^{2021} - 1)$ are coprime. Since the product $b \times (b^{2021} - 1) = b^{2022} - b = a^2$ is a perfect square, we get both factors b and $(b^{2021} - 1)$ must be perfect squares or the negatives of perfect squares, or one of them must be zero.

- If both b and $(b^{2021} - 1)$ are positive perfect squares, then let $b = x^2$ and $(b^{2021} - 1) = y^2$. Therefore $(x^2)^{2021} - 1 = y^2$ and thus

  $$1 = x^{4042} - y^2 = (x^{2021} - y)(x^{2021} + y).$$

  However since 1 is prime and the only integer factorizations of 1 are 1×1 and $(-1) \times (-1)$, we must have

  $$(x^{2021} - y) = (x^{2021} + y)$$

  Hence $y = 0$ which is a contradiction.

- If both b and $(b^{2021} - 1)$ are negative perfect squares, then let $b = -x^2$ and $(b^{2021} - 1) = -y^2$. Therefore $(-x^2)^{2021} - 1 = -y^2$ and thus

  $$x^{4042} + y^2 = 1.$$

  However the minimum value of $x^{4042} + y^2$ is $1 + 1 = 2$, so this is a contradiction too.

- If $b = 0$ then we get $a^2 + 0 = 0^{2022}$. So $a = 0$ and we get the solution $(a, b) = (0, 0)$.

- If $(b^{2021} - 1) = 0$ then we get $b = 1$. So $a^2 + 1 = 1^{2022}$. So $a = 0$ and we get the solution $(a, b) = (0, 1)$.

Therefore the only solutions for (a, b) are: $(0, 0)$ and $(0, 1)$.

2. **Problem:** Find all triples (a, b, c) of real numbers such that

$$a^2 + b^2 + c^2 = 1 \qquad \text{and} \qquad a(2b - 2a - c) \geq \frac{1}{2}.$$

Solution: (Viet Hoang)
The equations can be rewritten as

$$a^2 + b^2 + c^2 = 1 \qquad \text{and} \qquad 4ab - 4a^2 - 2ac \geq 1.$$

Substituting the first equation into the second and rearranging yields

$$\begin{aligned} 4ab - 4a^2 - 2ac &\geq a^2 + b^2 + c^2 \\ 5a^2 + b^2 + c^2 + 2ac - 4ab &\leq 0 \\ (2a - b)^2 + (a + c)^2 &\leq 0. \end{aligned}$$

Hence $b = 2a$ and $c = -a$. But $a^2 + b^2 + c^2 = 1$. So

$$a^2 + (2a)^2 + (-a)^2 = 1$$

and thus $a = \pm \frac{1}{\sqrt{6}}$. Therefore the final answers are:

$$(a, b, c) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right) \text{ and } \left(\frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right).$$

3. **Problem:** Let $\mathcal{S}$ be a set of 10 positive integers. Prove that one can find two disjoint subsets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ of $\mathcal{S}$ with $|A| = |B|$ such that the sums

$$x = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$$

and

$$y = \frac{1}{b_1} + \cdots + \frac{1}{b_k}$$

differ by less than 0.01; i.e., $|x - y| < 1/100$.

Solution: (Ishan Nath)
Partition the interval $(0.00, 2.50]$ into 250 intervals each of size 0.01.

$$(0.00, 2.50] = (0.00, 0.01] \cup (0.01, 0.02] \cup (0.02, 0.03] \cup \cdots \cup (2.49, 2.50].$$

Now consider all possible sets, S, we can choose from the given 10 positive integers with $|S| = 5$. Because each of the positive integers must be different, the smallest possible reciprocal sum of one of these sets is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} < 2.50.$$

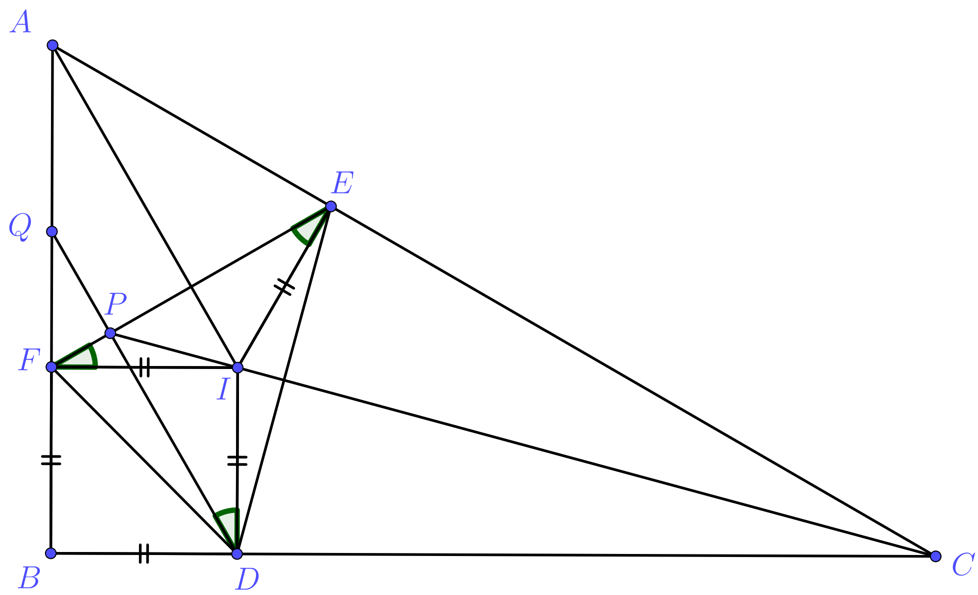
Therefore each of these different sets has a reciprocal sum lying somewhere in the interval $(0.00, 2.50)$. The number of such sets S is $\binom{10}{5} = 252$ but the number of intervals in our partition is only 250. By the pigeonhole principle there exists at least one interval, $(x, x + 0.01]$, and two distinct sets S_1, S_2 such that both reciprocal sums,

$$\sum_{s \in S_1} \frac{1}{s} \quad \text{and} \quad \sum_{s \in S_2} \frac{1}{s}$$

lie in $(x, x + 0.01]$. The reciprocal sums of S_1 and S_2 have difference less than $1/100$ because they both lie in the interval $(x, x + 0.01]$. If S_1 and S_2 are disjoint then we can simply choose $A = S_1$ and $B = S_2$. Otherwise let $C = S_1 \cap S_2$ be the intersection of S_1 and S_2 and then let $A = S_1 \setminus C$ and let $B = S_2 \setminus C$. The sets A and B are disjoint and equisized because $|S_1| = |S_2|$ and $S_1 \neq S_2$.

4. **Problem:** Triangle ABC is right-angled at B and has incentre I. Points D, E and F are the points where the incircle of the triangle touches the sides BC, AC and AB respectively. Lines CI and EF intersect at point P. Lines DP and AB intersect at point Q. Prove that $AQ = BF$.

Solution: (Kevin Shen)
First note that $ID = IE = IF$ because they are all radii of the incircle, and $\angle BFI = \angle BDI = 90°$ because tangents are perpendicular to radii. Since $\angle ABC = 90°$ we have $BFID$ a square and so $BD = BF = ID$ too. Thus $\triangle EIF$ is isosceles and so $\angle IFE = \angle FEI$.



Since CI is the bisector of $\angle DCE$, we see that D and E are reflections of each other over line CIP. Therefore $\angle IDP = \angle PEI$. Hence

$$\angle IDP = \angle FEI = \angle IFP$$

and therefore quadrilateral $FPID$ is cyclic. Therefore $\angle FPD = \angle FID = 90°$ ($BFID$ is a square). Since AI is the bisector of $\angle EAF$, we see that E and F are reflections of each other over line AI. Therefore $EF \perp AI$. Hence

$$AI||DQ$$

because they are both perpendicular to EF. We also have $AQ||ID$ (because $BFID$ is a square) so $QAID$ is a parallelogram. Therefore

$$AQ = ID = BF$$

as required.

5. **Problem:** The sequence $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 2022$ and $x_{n+1} = 7x_n + 5$ for all positive integers n. Determine the maximum positive integer m such that

$$\frac{x_n(x_n-1)(x_n-2)\ldots(x_n-m+1)}{m!}$$

is never a multiple of 7 for any positive integer n.

Solution A: (Ishan Nath)
We claim the answer is 404. First, we notice that $m \le 2022$. Otherwise,

$$\frac{x_1(x_1-1)\cdots(x_1-m+1)}{m!} = \frac{2022(2022-1)\cdots(2022-m+1)}{m!} = 0,$$

which is a multiple of 7. Then, since $x_n \ge x_1 = 2022$ for all n, we can write

$$\begin{aligned}\frac{x_n(x_n-1)\cdots(x_n-m+1)}{m!} &= \frac{x_n(x_n-1)\cdots(x_n-m+1)(x_n-m)(x_n-m-1)\cdots 1}{m! \times (x_n-m)(x_n-m-1)\cdots 1} \\ &= \frac{x_n!}{m!(x_n-m)!}.\end{aligned}$$

For a positive integer n, we define $\nu(n)$ as the exponent of 7 in the prime factorization of n. For example, $\nu(1) = 0$ and $\nu(98) = 2$. Note $\nu(ab) = \nu(a) + \nu(b)$ and $\nu(a/b) = \nu(a) - \nu(b)$. We prove the following two Lemmas:

- **Lemma 1:**

$$\nu(n!) = \sum_{i=1}^{d} \left\lfloor \frac{n}{7^i} \right\rfloor = \left\lfloor \frac{n}{7} \right\rfloor + \left\lfloor \frac{n}{49} \right\rfloor + \cdots + \left\lfloor \frac{n}{7^d} \right\rfloor,$$

  where d is the largest integer such that $7^d \le n$, for all non-negative integers n.

  Proof: Note that

$$\nu(n!) = \nu(1) + \nu(2) + \cdots + \nu(n).$$

  $\nu(n!)$ has a contribution of $+1$ for each multiple of seven (less than or equal to n) and there are $\lfloor n/7 \rfloor$ such numbers. Each multiple of 49 contributes another $+1$ to $\nu(n!)$, and there are $\lfloor n/49 \rfloor$ such numbers. Generally, there is an additional $+1$ contributed for each multiple of 7^i, of which there are $\lfloor n/7^i \rfloor$ numbers. Adding up all these contributions, we get the desired result.

- **Lemma 2:** $\lfloor x \rfloor - \lfloor y \rfloor - \lfloor x - y \rfloor \ge 0$, for all reals x and y, with equality if and only if $\{x\} \ge \{y\}$.

  Proof: Let $x = \lfloor x \rfloor + \{x\}$ and $y = \lfloor y \rfloor + \{y\}$. Then $x - y = \lfloor x \rfloor - \lfloor y \rfloor + \{x\} - \{y\}$. Since $0 < \{x\}, \{y\} < 1$, we have $-1 < \{x\} - \{y\} < 1$.

  (a) If $0 \le \{x\} - \{y\} < 1$, i.e. $\{x\} \ge \{y\}$, then $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor$, so $\lfloor x \rfloor = \lfloor y \rfloor + \lfloor x - y \rfloor$.

  (b) Otherwise, $\lfloor x - y \rfloor = \lfloor x \rfloor - \lfloor y \rfloor - 1$, so $\lfloor x \rfloor > \lfloor x \rfloor - 1 = \lfloor y \rfloor + \lfloor x - y \rfloor$.

Now we can use Lemma 1 to compute

$$\begin{aligned}
\nu\left(\frac{x_n!}{m!(x_n-m)!}\right) &= \nu(x_n!) - \nu(m!) - \nu((x_n-m)!) \\
&= \sum_{i=1}^{d}\left\lfloor \frac{x_n}{7^i}\right\rfloor - \sum_{i=1}^{l_1}\left\lfloor \frac{m}{7^i}\right\rfloor - \sum_{i=1}^{l_2}\left\lfloor \frac{x_n-m}{7^i}\right\rfloor \\
&= \sum_{i=1}^{d}\left\lfloor \frac{x_n}{7^i}\right\rfloor - \sum_{i=1}^{d}\left\lfloor \frac{m}{7^i}\right\rfloor - \sum_{i=1}^{d}\left\lfloor \frac{x_n-m}{7^i}\right\rfloor \\
&= \sum_{i=1}^{d}\left(\left\lfloor \frac{x_n}{7^i}\right\rfloor - \left\lfloor \frac{m}{7^i}\right\rfloor - \left\lfloor \frac{x_n-m}{7^i}\right\rfloor\right).
\end{aligned}$$

Here d is the largest integer such that $7^d \leq x_n$, l_1 is the largest integer such that $7^{l_1} \leq m$, and l_2 is the largest integer such that $7^{l_2} \leq x_n - m$. Increasing the range of the sums does not affect the result, as we are simply adding terms of the form $\lfloor a/7^b \rfloor$, where $7^b > a$, which gives 0.

If we let $x = x_n/7^i$ and $y = m/7^i$, then this final sum consists of terms of the form $\lfloor x\rfloor - \lfloor y\rfloor - \lfloor x-y\rfloor \geq 0$. Therefore, we get that

$$\begin{aligned}
&7 \text{ doesn't divide } \frac{x_n(x_n-1)\cdots(x_n-m+1)}{m!} \\
&\text{if and only if } \nu\left(\frac{x_n!}{m!(x_n-m)!}\right) = 0 \\
&\text{if and only if } \left\lfloor \frac{x_n}{7^i}\right\rfloor = \left\lfloor \frac{m}{7^i}\right\rfloor + \left\lfloor \frac{x_n-m}{7^i}\right\rfloor \qquad \text{for all } 0 \leq i \leq d \\
&\text{if and only if } \left\{\frac{x_n}{7^i}\right\} \geq \left\{\frac{m}{7^i}\right\} \qquad \text{for all } 0 \leq i \leq d.
\end{aligned}$$

by Lemma 2. This must hold for all n. Notice

$$\left\{\frac{x_n}{7^i}\right\} \geq \left\{\frac{m}{7^i}\right\} \text{ if and only if } 7^i\left\{\frac{x_n}{7^i}\right\} \geq 7^i\left\{\frac{m}{7^i}\right\},$$

and $7^b\{a/7^b\}$ is simply the remainder of a modulo 7^b. Hence we have

$$\left\{\frac{x_n}{7^i}\right\} \geq \left\{\frac{m}{7^i}\right\} \text{ if and only if } x_n \pmod{7^i} \geq m \pmod{7^i}.$$

Since $x_1 = 2022 = 5\cdot 7^3 + 6\cdot 7^2 + 1\cdot 7^1 + 6\cdot 7^0$, we inductively get

$$x_n = 5\cdot 7^{n+2} + 6\cdot 7^{n+1} + 1\cdot 7^n + 6\cdot 7^{n-1} + 5\cdot 7^{n-2} + \cdots + 5\cdot 7^0.$$

Using this, we can find the smallest value of $x_n \pmod{7^i}$:

- For $i = 1$, the smallest value is $5\cdot 7^0$, when $n \geq 2$.
- For $i = 2$, the smallest value is $1\cdot 7^1 + 6\cdot 7^0$, when $n = 1$.
- For $i = 3$, the smallest value is $1\cdot 7^2 + 6\cdot 7^1 + 5\cdot 7^0$, when $n = 2$.
- For $i = 4$, the smallest value is $1\cdot 7^3 + 6\cdot 7^2 + 5\cdot 7^1 + 5\cdot 7^0$, when $n = 3$.
- For $i \geq 5$, the smallest value is $5\cdot 7^3 + 6\cdot 7^2 + 1\cdot 7^1 + 6\cdot 7^0$, when $n = 1$.

Thus if we write m in the form $m = a_3 \cdot 7^3 + a_2 \cdot 7^2 + a_1 \cdot 7^1 + a_0 \cdot 7^0$, where $0 \le a_i \le 6$, we must have $a_0 \le 5$, $a_1 \le 1$, $a_2 \le 1$, and $a_3 \le 1$, which are necessary and sufficient.

Therefore the maximum integer m is achieved when $a_0 = 5$ and $a_1 = a_2 = a_3 = 1$. This gives $m = 7^3 + 7^2 + 7^1 + 5 = 404$.

Solution B: (Ishan Nath)
As in solution A, we make the observation that

$$\frac{x_n(x_n-1)\cdots(x_n-m+1)}{m!} = \binom{x_n}{m}.$$

- **Lemma:** Let a and b be two positive integers with $\overline{a_k a_{k-1}\cdots a_0}$ and $\overline{b_k b_{k-1}\cdots b_0}$ the base 7 representations of a and b respectively, possibly with leading zeroes. $\binom{a}{b}$ is not a multiple of 7 if and only if $a_i \ge b_i$ for all $0 \le i \le k$.

  Proof: By Lucas' Theorem

  $$\binom{a}{b} \equiv \prod_{i=0}^{k} \binom{a_k}{b_k} \pmod 7.$$

  In particular, we have

  $$\binom{a}{b} \equiv 0 \pmod 7 \text{ if and only if } \binom{a_k}{b_k} \equiv 0 \pmod 7 \text{ for some } k$$
  $$\text{if and only if } a_k < b_k \text{ for some } k.$$

  □

Now, note $x_1 = 2022 = \overline{5616}$ in base 7, and thus $x_n = \overline{5616555\cdots 55}$ in base 7 (where the rightmost $n-1$ digits are '5's). Let $x_n = \overline{a_k a_{k-1}\cdots a_0}$ in base 7. Hence if $m = \overline{m_k m_{k-1}\cdots m_0}$, we get that

$$7 \nmid \binom{x_n}{m} \text{ for all } n \text{ if and only if } a_k \ge m_k \text{ for all } n, k.$$

When $n = 1$ we have $a_k = 0$ for $k \ge 4$, so this means $m_k = 0$ for $k \ge 4$. Also notice that

- $a_0 \ge 5$ with equality when $n \ge 2$,
- $a_1 \ge 1$ with equality when $n = 1$,
- $a_2 \ge 1$ with equality when $n = 2$, and
- $a_3 \ge 1$ with equality when $n = 3$.

This implies $m_0 \le 5$ and $m_1, m_2, m_3 \le 1$, which gives necessary and sufficient conditions for m.

The maximum value of m can now be found by taking the maximum values of all m_k. This gives $m = \overline{1115}$ in base 7, or $m = 404$ (in base 10).