New Zealand Mathematical Olympiad Committee

NZMO Round Two 2022 — Instructions

Exam date: 17^{th} September

General Instructions

1. You have 3 hours to work on the exam.
2. There are 5 problems, each worth equal marks. You should attempt all 5 problems. You may work on them in any order.
3. Geometrical instruments (ruler and compasses) may be used. Calculators, phones, computers and electronic devices of any sort are not permitted.
4. Write your solutions on blank paper. There is no limit to how many answer pages you may use. Write your name and problem number at the top of every page as you start using it.
5. Full written solutions — not just answers — are required, with complete proofs of any assertions you make. Your marks will depend on the clarity of your mathematical presentation. Work in rough first, and then draft a neat final version of your best attempt.
6. **All** pages, including your rough work, are to be submitted.
7. The contest problems are to be kept confidential until they are posted on the NZMOC webpage www.mathsolympiad.org.nz. Any discussion of the problems before this has occured is strictly forbidden.

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NZMO Round Two 2022 — Problems

Exam date: 17^{th} September

- There are 5 problems. You should attempt to find solutions for as many as you can.
- Solutions (that is, answers with justifications) and not just answers are required for all problems, even if they are phrased in a way that only asks for an answer.
- Read and follow the "General instructions" accompanying these problems.

Problems

1. Find all integers a, b such that
$$a^2 + b = b^{2022}.$$

2. Find all triples (a, b, c) of real numbers such that
$$a^2 + b^2 + c^2 = 1 \qquad \text{and} \qquad a(2b - 2a - c) \geq \frac{1}{2}.$$

3. Let $\mathcal{S}$ be a set of 10 positive integers. Prove that one can find two disjoint subsets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ of $\mathcal{S}$ with $|A| = |B|$ such that the sums
$$x = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$$
and
$$y = \frac{1}{b_1} + \cdots + \frac{1}{b_k}$$
differ by less than 0.01; i.e., $|x - y| < 1/100$.

4. Triangle ABC is right-angled at B and has incentre I. Points D, E and F are the points where the incircle of the triangle touches the sides BC, AC and AB respectively. Lines CI and EF intersect at point P. Lines DP and AB intersect at point Q. Prove that $AQ = BF$.

5. The sequence $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 2022$ and $x_{n+1} = 7x_n + 5$ for all positive integers n. Determine the maximum positive integer m such that
$$\frac{x_n(x_n - 1)(x_n - 2)\ldots(x_n - m + 1)}{m!}$$
is never a multiple of 7 for any positive integer n.