New Zealand Mathematical Olympiad Committee

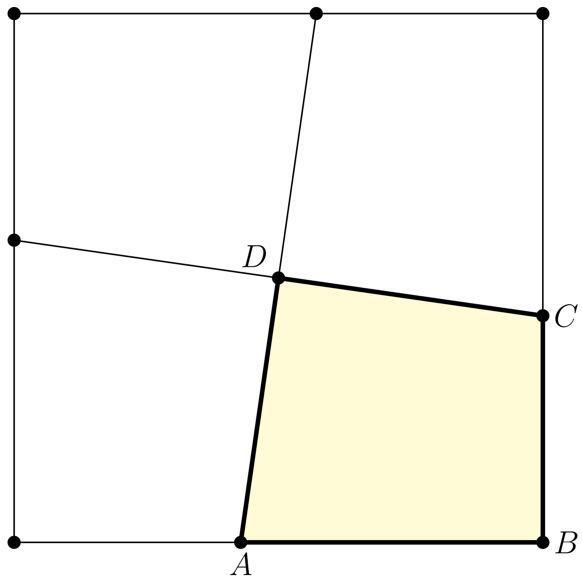
NZMO Round Two 2021 — Solutions

1. **Problem:** Let $ABCD$ be a convex quadrilateral such that $AB + BC = 2021$ and $AD = CD$. We are also given that

$$\angle ABC = \angle CDA = 90^\circ$$

Determine the length of the diagonal BD.

Solution: Since $AD = DC$ and $\angle ADC = 90^\circ$, we can fit four copies of quadrilateral $ABCD$ around vertex D as shown in the diagram.



The outer shape is a quadrilateral because $\angle DAB + \angle BCD = 180^\circ$. Moreover it is a rectangle because $\angle ABC = 90^\circ$. In fact it is a square with side-length 2021 because of rotational symmetry and $AB + BC = 2021$. Also D is the centre of the square because it is the centre of the rotational symmetry. So BD is the distance from a vertex to the centre of the square, which is half the length of the diagonal of the square. Thus

$$BD = \frac{1}{2}\left(2021\sqrt{2}\right) = \frac{2021}{\sqrt{2}}.$$

Alternative Solution:

First let $x = AD = DC$ and $a = BD$ and $y = AB$ and $z = BC$. Now initially we can apply Pythagoras in triangles CDA and ABC to get $x^2 + x^2 = AC^2$ and $y^2 + z^2 = AC^2$ respectively. Putting this together gives us

$$x^2 = \frac{y^2 + z^2}{2}.$$

Now note that the opposite angles $\angle ABC$ and $\angle CDA$ (in quad $ABCD$) are supplimentary. Therefore $ABCD$ is a cyclic quadrilateral. Equal chords subtend equal arcs (and chords $AD = DC$ are equal) so $\angle ABD = \angle DBC$. Furthermore, since $\angle ABC$ is a right angle, this means that $\angle ABD = \angle DBC = 45°$.

For any three points P, Q and R, let $|PQR|$ denote the area of triangle PQR. Now consider the total area of quadrilateral $ABCD$ calculated in two ways:

$$|ABC| + |CDA| = |ABD| + |DBC|.$$

We calculate the areas of the right-angled triangles using the $\triangle = \frac{bh}{2}$ formula, and we calculate the area of the 45°-angled triangles using the $\triangle = \frac{1}{2}ab\sin C$ formula.

$$\frac{yz}{2} + \frac{x^2}{2} = \tfrac{1}{2}ay\sin(45°) + \tfrac{1}{2}az\sin(45°)$$

At this point we can substitute $x^2 = \frac{1}{2}(y^2 + z^2)$ into this equation, and rearrange:

$$\begin{aligned}
\frac{yz}{2} + \frac{x^2}{2} &= \frac{ay\sin(45°)}{2} + \frac{az\sin(45°)}{2} \\
\frac{yz}{2} + \frac{y^2 + z^2}{4} &= \frac{ay}{2\sqrt{2}} + \frac{az}{2\sqrt{2}} \\
\frac{2yz + y^2 + z^2}{4} &= \frac{ay + az}{2\sqrt{2}} \\
\frac{(y+z)^2}{4} &= \frac{a(y+z)}{2\sqrt{2}} \\
\frac{y+z}{\sqrt{2}} &= a.
\end{aligned}$$

Finally since $y + z = 2021$ this gives our final answer of $BD = a = \frac{2021}{\sqrt{2}}$.

2. **Problem:** Prove that

$$x^2 + \frac{8}{xy} + y^2 \geq 8.$$

for all positive real numbers x and y.

Solution: Since square numbers are always non-negative we have

$$(x-y)^2 \geq 0 \qquad \text{and} \qquad (xy-2)^2 \geq 0.$$

Also since x and y are positive we have $\frac{2}{xy} > 0$. Combining this all together gives us:

$$(x-y)^2 + \frac{2}{xy}(xy-2)^2 \geq 0.$$

From here we expand and simplify:

$$\begin{aligned} (x^2 - 2xy + y^2) + \frac{2}{xy}(x^2y^2 - 4xy + 4) &\geq 0 \\ x^2 - 2xy + y^2 + 2xy - 8 + \frac{8}{xy} &\geq 0 \\ x^2 + \frac{8}{xy} + y^2 &\geq 8 \end{aligned}$$

as required.

Alternative Solution:
Consider the AM-GM inequality applied to $\left\{x^2, \frac{4}{xy}, \frac{4}{xy}, y^2\right\}$.

$$\begin{aligned} \frac{x^2 + \frac{4}{xy} + \frac{4}{xy} + y^2}{4} &\geq \sqrt[4]{x^2 \times \frac{4}{xy} \times \frac{4}{xy} \times y^2} \\ \frac{x^2 + \frac{8}{xy} + y^2}{4} &\geq 2 \\ x^2 + \frac{8}{xy} + y^2 &\geq 8. \end{aligned}$$

3. **Problem:** Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be a set of n distinct positive integers, such that the sum of any 3 of them is a prime number. What is the maximum value of n?

Solution: First we show that $n = 4$ is possible with an example. The example $\{x_1, x_2, x_3, x_4\} = \{1, 3, 7, 9\}$ satisfies the problem because:

- $1 + 3 + 7 = 11$ is prime,
- $1 + 3 + 9 = 13$ is prime,
- $1 + 7 + 9 = 17$ is prime, and
- $3 + 7 + 9 = 19$ is prime.

We still have to prove that $n \geq 5$ is impossible.

Consider any set $\{x_1, x_2, x_3, \ldots, x_n\}$ such that the sum of any 3 of them is a prime number. Also consider the three "pigeonholes" modulo 3; the residue classes 0, 1 and 2. If all three pigeonholes were non-empty, then it would be possible to choose three numbers – one from each pigeonhole. This would result in a sum which is $0 + 1 + 2 \equiv 0 \pmod 3$, and since the numbers are distinct positive integers, this sum would be > 3. Thus the sum would not be prime which is a contradiction. Hence at least one of the pigeonholes must be empty. *i.e.*

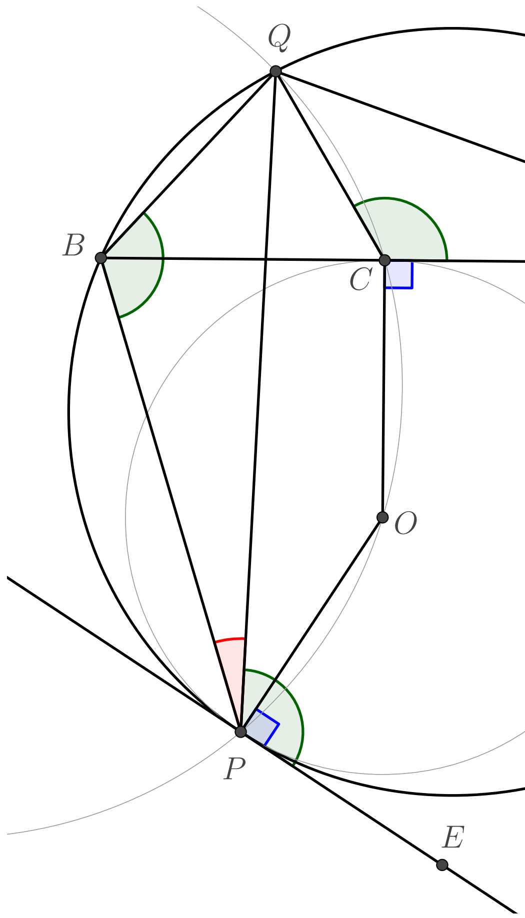
The numbers $\{x_1, \ldots, x_n\}$ are distributed amongst
(at most) two different residue classes modulo 3.

Now assume for the sake of contradiction that $n \geq 5$. By the pigeonhole principle at least one residue class contains at least 3 of the numbers. The sum of any three numbers from the same residue class is always a multiple of 3 and so this is a contradiction.
Therefore $n < 5$ as required.

Comment: The example $\{1, 3, 7, 9\}$ is not the only example that satisfies the problem with $n = 4$. Here are many other examples: $\{1, 5, 7, 11\}$, $\{3, 5, 11, 15\}$, $\{1, 3, 13, 15\}$, $\{3, 9, 11, 17\}$, $\{5, 9, 15, 17\}$, … Searching for an example when $n = 4$ is much easier if you conjecture that all the x_i must be odd.

4. **Problem:** Let AB be a chord of circle Γ. Let O be the centre of a circle which is tangent to AB at C and internally tangent to Γ at P. Point C lies between A and B. Let the circumcircle of triangle POC intersect Γ at distinct points P and Q. Prove that $\angle AQP = \angle CQB$.

   Solution: Construct the tangent line to Γ at P. Note that this line is also tangent to the circle through points C and P with centre O. Also construct point E on this tangent line to the right of P. Note that $\angle EPO = 90°$ and $\angle OCA = 90°$ because the radii and tangents are perpendicular.



Let $x = \angle PBQ$

$$\angle EPQ = x \quad \text{(by alternate segment theorem)}$$

$$\angle OPQ = x - 90° \quad \text{(because } \angle EPO = 90°\text{)}$$

$$\angle QCO = 180° - \angle OPQ \quad \text{(because opposite angles in a cyclic quad are supplimentary)}$$

$$= 270° - x$$

$$\angle ACQ = 360° - \angle QCO - \angle OCA \quad \text{(angles around point } C \text{ are } 360°\text{)}$$

$$= 360° - (270° - x) - 90°$$

$$= x.$$

We also have $\angle QPB = \angle QAB$ (angles subtended by chord QB) in cyclic quad $QAPB$. Therefore we have similar triangles

$$\triangle QBP \sim \triangle QCA \qquad (\angle PBQ = \angle ACQ \text{ and } \angle QPB = \angle QAC)$$

Hence $\angle AQC = \angle PQB$. Therefore

$$\angle AQP = \angle AQC + \angle CQP = \angle PQB + \angle CQP = \angle CQB.$$

5. **Problem:** Find all pairs of integers x, y such that

$$y^5 + 2xy = x^2 + 2y^4.$$

Solution: Rearrange and factorize to get

$$y^2(y-1)(y^2-y-1) = (x-y)^2.$$

Note that y and $(y-1)$ are coprime (their greatest common divisor is 1) because they are consecutive integers. Note since $y(y-1)$ and (y^2-y-1) are consecutive integers, we see that (y^2-y-1) is coprime to both y and $(y-1)$. Therefore the three factors

$$y^2, (y-1) \text{ and } (y^2-y-1) \quad \text{are pairwise coprime.}$$

Since their product is a perfect square it follows that either: one of y^2, $(y-1)$ and (y^2-y-1) is zero, or all three of them are perfect squares. So we have four cases:

- **Case A:** $y = 0$.
  Substituting this into the original equation yields $(0)^5 + 2x(0) = x^2 + 2(0)^4$. Solving this quadratic yields $x = 1$ and so $(x, y) = (0, 0)$ is the only solution in this case.
- **Case B:** $y = 1$.
  Substituting this into the original equation yields $(1)^5 + 2x(1) = x^2 + 2(1)^4$. Solving this quadratic yields $x = 1$ and so $(x, y) = (1, 1)$ is the only solution in this case.
- **Case C:** $y^2 - y - 1 < 0$.
  This rearranges to give us $(2y-1)^2 < 5$. But the only odd square less than 5 is 1, and so we would have $(2y-1) = \pm 1$. which leads to $y = 0, 1$ (but we have already covered this in Cases A and B).
- **Case D:** $y^2 - y - 1 = k^2$.
  - If $y > 2$ then $(y-1)^2 = y^2 - 2y + 1 < y^2 - y - 1 < y^2$. This would give us $(y-1)^2 < k^2 < y^2$ which is a contradiction because $(y-1)^2$ and y^2 are consecutive squares.
  - If $y < -1$ then $(-y)^2 < y^2 - y - 1 < y^2 - 2y + 1 = (-y+1)^2$. This would give us $y^2 < k^2 < (y-1)^2$ which is a contradiction because $(-y)^2$ and $(-y+1)^2$ are consecutive squares.

  Therefore we must have $-1 \leq y \leq 2$. We have already covered $y = 0$ and $y = 1$ in cases A and B respectively. So it suffices now only to consider $y = -1$ and $y = 2$.

  - If $y = -1$ then $(-1)^5 + 2x(-1) = x^2 + 2(-1)^4$. This rearranges into $(x+1)^2 = -2$ which has no real solutions.
  - If $y = 2$ then $2^5 + 4x = x^2 + 2 \times 2^4$. This rearranges to give $x^2 - 4x = 0$ which has solutions $x = 0$ and $x = 4$.

  Hence in this case we have solutions $(x, y) = (0, 2)$ and $(4, 2)$.

In summary we have four distinct solutions for (x, y) being:

$$(x, y) = (0, 0), (1, 1), (0, 2) \text{ and } (4, 2).$$

Alternative Solution:

In a similar manner to above get to Case D:

$$y^2 - y - 1 = k^2.$$

Then rearrange it to give $(2y-1)^2 - 4k^2 = 5$. This then factorizes as a difference between two squares as

$$(2y - 1 + 2k)(2y - 1 - 2k) = 5$$

Since 5 is prime it can only be factored in two ways: $5 = 5 \times 1 = (-5) \times (-1)$. The sum of these two factors is $(2y - 1 + 2k) + (2y - 1 - 2k) = 4y - 2$. Therefore:

$$4y - 2 = 5 + 1 \qquad \text{or} \qquad 4y - 2 = (-5) + (-1).$$

From $(4y - 2) = 6$ we get $y = 2$, and from $(4y - 2) = -6$ we get $y = -1$.

Thus we have ruled out all possibilities except for $y = -1, 0, 1, 2$. Checking these each individually yields the four answers.

- $y = 2$ yields $(2)^5 + 2x(2) = x^2 + 2(2)^4$ which simplifies to become $x^2 - 4x = 0$, and has solutions $x = 0$ and $x = 4$.
- $y = 1$ yields $(1)^5 + 2x(1) = x^2 + 2(1)^4$ which simplifies to become $x^2 - 2x + 1 = 0$, and has solution $x = 1$ only.
- $y = 0$ yields $(0)^5 + 2x(0) = x^2 + 2(0)^4$ which simplifies to become $x^2 = 0$, and has solution $x = 0$ only.
- $y = -1$ yields $(-1)^5 + 2x(-1) = x^2 + 2(-1)^4$ which simplifies to $x^2 + 2x + 3 = 0$, but this has no real solutions.

Thus all solutions are

$$(x, y) = (0, 2), (4, 2), (1, 1) \text{ and } (0, 0).$$