New Zealand Mathematical Olympiad Committee

NZMO Round Two 2020 — Solutions

1. **Problem:** Let $P(x) = x^3 - 2x + 1$ and let $Q(x) = x^3 - 4x^2 + 4x - 1$. Show that

$$\text{if } P(r) = 0 \text{ then } Q(r^2) = 0.$$

Solution: Notice that $P(x) = x^3 - 2x + 1 = (x-1)(x^2+x-1)$, and the roots of $x^2 + x - 1$ are $\frac{-1\pm\sqrt{5}}{2}$. Therefore the roots of $P(x)$ are

$$1\,,\ \frac{\sqrt{5}-1}{2} \text{ and } \frac{-1-\sqrt{5}}{2}.$$

Notice also that $Q(x) = x^3 - 4x^2 + 4x - 1 = (x-1)(x^2-3x+1)$, and the roots of $x^2 - 3x + 1$ are $\frac{3\pm\sqrt{5}}{2}$. Therefore the roots of $Q(x)$ are

$$1\,,\ \frac{3+\sqrt{5}}{2} \text{ and } \frac{3-\sqrt{5}}{2}.$$

So it suffices to check that $1^2 = 1$ and that $\left(\frac{-1\pm\sqrt{5}}{2}\right)^2 = \frac{1\pm2\sqrt{5}+5}{4} = \frac{3\pm\sqrt{5}}{2}$.

Alternative Solution:
First notice that

$$\begin{aligned}(x^3-2x-1)P(x) &= (x^3-2x-1)(x^3-2x+1)\\ &= x^6 - 4x^4 + 4x^2 - 1\\ &= Q(x^2).\end{aligned}$$

If r is any root of $P(x)$ then $P(r) = 0$. This implies that

$$Q(r^2) = (r^3 - 2r + 1)P(r) = (r^3 - 2r + 1) \times 0 = 0.$$

Hence r^2 is a root of $Q(x)$.

2. **Problem:** Find the smallest positive integer N satisfying the following three properties.

- N leaves a remainder of 5 when divided by 7.
- N leaves a remainder of 6 when divided by 8.
- N leaves a remainder of 7 when divided by 9.

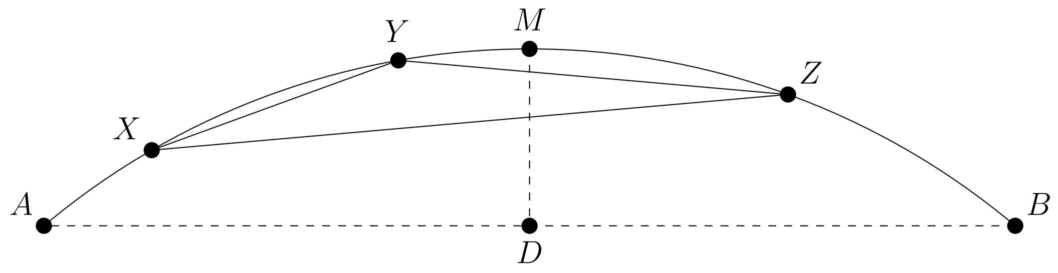
Solution: We notice that $\{5, 6, 7\}$ are each 2 less than $\{7, 8, 9\}$ respectively. Therefore $N+2$ must be a multiple of 7, 8 and 9. Since 7, 8 and 9 are pairwise coprime, this means that

$$(N+2) \text{ is a multiple of } 7 \times 8 \times 9 = 504.$$

Therefore the smallest positive possibility for $N + 2$ is 504. Thus $N = 502$.

3. **Problem:** There are 13 marked points on the circumference of a circle with radius 13. Prove that we can choose three of the marked points which form a triangle with area less than 13.

Solution: Divide the circle into 6 equal ($60°$) arcs. By the pigeon-hole principle (since $13 > 2\times 6$) there exists at least one arc which contains at least three of the marked points. Let this $60°$ arc be AB, and let the three marked points be X, Y and Z in that order.



Let M be the midpoint of arc AB. Points X and Z both lie within the minor arc AB so the base of triangle $\triangle XYZ$ is less than or equal to the base of triangle $\triangle AMB$. Also the distance from Y to XZ is less than or equal to the distance from Y to AB, which is at most the distance from M to AB. Hence: the base and height of triangle $\triangle XYZ$ are less than or equal to the base and height of triangle $\triangle AMB$ respectively. Therefore

$$\text{area}(\triangle XYZ) \leq \text{area}(\triangle AMB).$$

So it suffices to show that $\text{area}(\triangle AMB) < 13$. Let O be the centre of the circle and let D be the foot of the altitude from M to AB. Note that $\triangle ABO$ is equilateral and D is the midpoint of AB so $AD = \frac{13}{2}$. Furthermore, by Pythagoras in $\triangle ADO$ we get

$$OD = \sqrt{OA^2 - AD^2} = \sqrt{13^2 - \left(\frac{13}{2}\right)^2} = \frac{13}{2}\sqrt{3}.$$

$$\implies MD = MO - OD = 13 - \frac{13}{2}\sqrt{3} = 13\left(1 - \frac{\sqrt{3}}{2}\right).$$

Now we can calculate the area of triangle $\triangle AMB$. The base is $AB = 13$ because ABO is equilateral. The height is $MD = 13\left(1 - \frac{\sqrt{3}}{2}\right)$. Therefore

$$\text{area}(\triangle AMB) = \frac{13 \times 13\left(1 - \frac{\sqrt{3}}{2}\right)}{2} = \frac{169(2-\sqrt{3})}{4}.$$

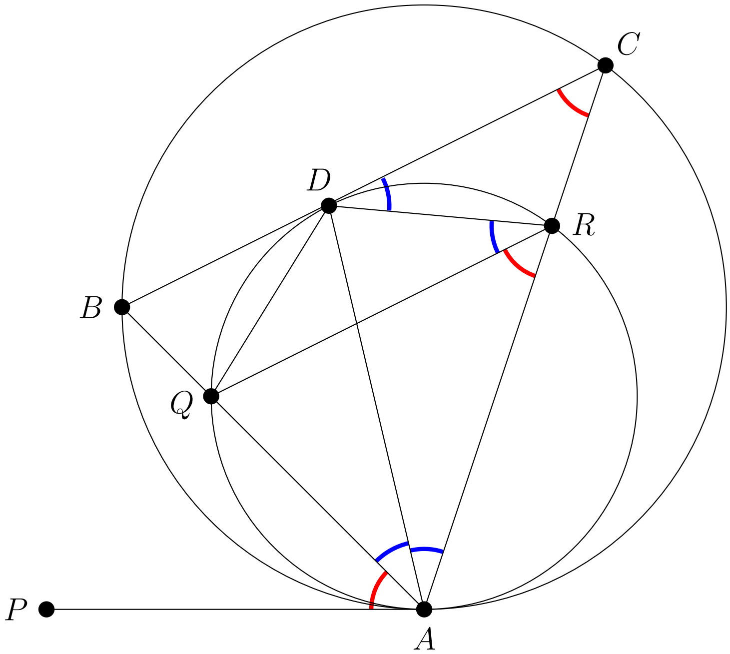
And we can confirm that $\frac{169(2-\sqrt{3})}{4} < 13$ because

$$\begin{aligned} & \frac{169(2-\sqrt{3})}{4} < 13 \\ \iff\quad & 13(2-\sqrt{3}) < 4 \\ \iff\quad & 26 - 13\sqrt{3} < 4 \\ \iff\quad & 22 < 13\sqrt{3} \\ \iff\quad & 22^2 < 13^2 \times 3 \\ \iff\quad & 484 < 507. \end{aligned}$$

as required.

4. **Problem:** Let Γ_1 and Γ_2 be circles internally tangent at point A, with Γ_1 inside Γ_2. Let BC be a chord of Γ_2 which is tangent to Γ_1 at point D. Prove that line AD is the angle bisector of $\angle BAC$.

   Solution: Let λ be the common tangent of Γ_1 and Γ_2 at point A. Let P be a point on λ such that P and C are on opposite sides of line AB. Let Q and R be the points of intersection of Γ_1 with AB and AC respectively.



$$\begin{aligned} \angle BCA &= \angle BAP && \text{(alternate segment theorem in } \Gamma_2\text{)} \\ &= \angle QAP \\ &= \angle QRA && \text{(alternate segment theorem in } \Gamma_1\text{)} \end{aligned}$$

   Therefore lines BC and QR are parallel.
   Now consider $\angle BAD$.

$$\begin{aligned} \angle BAD &= \angle QAD \\ &= \angle QRD && (QARD \text{ is cyclic}) \\ &= \angle CDR && (QR \parallel BC) \\ &= \angle DAR && \text{(alternate segment theorem)} \\ &= \angle DAC. \end{aligned}$$

   Since $\angle BAD = \angle DAC$, we are done.

   Alternative Solution:
   Consider the dilation centered at A which sends Γ_1 to Γ_2. This dilation sends point D to the point $D' \in \Gamma_2$ such that points A, D and D' are colinear. This dilation also sends line BC to the line tangent to Γ_2 at point D'. Therefore the tangent to Γ_2 at point D' is parallel to chord BC. Hence D' is the midpoint of arc BC. *I.e.* BD' and $D'C$ have the same arc-length. Since equal arcs subtend equal angles, we deduce that $\angle BAD' = \angle D'AC$ as required.

5. **Problem:** A sequence of As and Bs is called *antipalindromic* if writing it backwards, then turning all the As into Bs and vice versa, produces the original sequence. For example $ABBAAB$ is antipalindromic. For any sequence of As and Bs we define the *cost* of the sequence to be the product of the positions of the As. For example, the string $ABBAAB$ has cost $1 \cdot 4 \cdot 5 = 20$. Find the sum of the costs of all antipalindromic sequences of length 2020.

   Solution: For each integer $0 \le k \le 1009$ define a k-pal to be any sequence of 2020 As and Bs, where the first k terms are B, the last k terms are B, and the middle $(2020-2k)$ terms form an antipalindromic sequence.
   Now for any k, define $f(k)$ to be sum of the costs of all k-pals. Note that any k-pal can be created from a $(k+1)$-pal by either

   - (A) replacing the B in position $(k+1)$ with an A, or
   - (B) replacing the B in position $(2021-k)$ with an A.

   Therefore the sum of the costs of all k-pals formed using operation (A) is $(k+1) \times f(k+1)$. Similarly the sum of the costs of all k-pals formed using operation (B) is $(2021-k) \times f(k+1)$. Hence

   $$f(k) = (k+1)f(k+1) + (2020-k)f(k+1) = ((k+1)+(2020-k))f(k+1) = 2021f(k+1).$$

   Now we note that there are two different 1009-pals, with costs equal to 1010 and 1011 respectively. So

   $$f(1009) = 1010 + 1011 = 2021.$$

   Now if we use the formula $f(k) = 2021f(k+1)$ iteratively, we get $f(1010-i) = 2021^i$ for each $i = 1, 2, 3, \ldots$ Therefore

   $$f(0) = 2021^{1010}$$

   which is our final answer.

   Alternative Solution:
   Let n be a positive integer. We will find an expression (in terms of n) for the sum of the costs of all antipalindromes of length $2n$. Note that a string of As and Bs of length $2n$ is an antipalindrome if and only if for each i, exactly one of the i^{th} and $(2n+1-i)^{\text{th}}$ letters is an A (and the other is a B).
   Let $x_1, x_2, \ldots, x_{2n}$ be variables. For any $1 \le a(1) < a(2) < \cdots < a(k) \le 2n$, consider the string of As and Bs of length $2n$, such that the $a(j)^{\text{th}}$ letter is A for all j (and all the other letters are B). Let this string correspond to the term $t = x_{a(1)}x_{a(2)}x_{a(3)} \ldots x_{a(k)}$. If $x_i = i$ for all i then the value of t is equal to the cost of it's corresponding string. Now consider the expression

   $$y = (x_1 + x_{2n})(x_2 + x_{2n-1}) \cdots (x_n + x_{n+1}) = \prod_{j=1}^{n}(x_j + x_{2n+1-j}).$$

   If we expand the brackets then we get 2^n terms, each in the form $t = x_{a(1)}x_{a(2)}x_{a(3)} \ldots x_{a(n)}$ such that for each $j = 1, 2, \ldots, n$ either $a(j) = j$ or $a(j) = 2n+1-j$. Therefore y is the sum of all terms that correspond to antipalindromes. Hence if we substitute $x_i = i$ for all i, then the value of y would be the sum of the costs of all antipalindromes. So the final answer is:

   $$\prod_{j=1}^{n}(j + (2n+1-j)) = \prod_{j=1}^{n}(2n+1) = (2n+1)^n.$$

Alternative Solution 2:
Let n be a positive integer. We will find an expression (in terms of n) for the sum of the costs of all antipalindromes of length $2n$. Let $\mathcal{P}$ denote the set of all antipalindromes of length $2n$, and let P be an antipalindrome chosen uniformly from $\mathcal{P}$. Note that for each $j = 1, 2, \ldots, n$ the j^{th} and $(2n+1-j)^{\text{th}}$ must be an A and a B in some order. Let X_j be the random variable defined by:

- $X_j = j$ if the j^{th} letter of P is an A and the $(2n+1-j)^{\text{th}}$ letter is a B.
- $X_j = 2n+1-j$ if the j^{th} letter of P is a B and the $(2n+1-j)^{\text{th}}$ letter is an A.

Notice that the cost of P is given by the product $X_1X_2X_3 \cdots X_n$. Now consider $f_j : \mathcal{P} \to \mathcal{P}$ to be the function which swaps the j^{th} and $(2n+1-j)^{\text{th}}$ letters of the string. Notice that f_j is a bijection that toggles the value of X_j. This means that X_j is equal to j or $(2n+1-j)$ with equal probabilities. Therefore

$$\mathbb{P}(X_j = j) = \mathbb{P}(X_j = 2n+1-j) = \frac{1}{2}.$$

Furthermore f_j preserves the value of X_i for all $i \neq j$. Therefore the variables X_i and X_j are independent. Therefore the expected value of the cost of P is given by:

$$\begin{aligned}
\mathbb{E}[\text{cost}(P)] &= \mathbb{E}\left[\prod_{i=1}^{n} X_i\right] \\
&= \prod_{i=1}^{n} \mathbb{E}\left[X_i\right] \\
&= \prod_{i=1}^{n} \left(\frac{1}{2}(i) + \frac{1}{2}(2n+1-i)\right) \\
&= \prod_{i=1}^{n} \frac{2n+1}{2} \\
&= \left(\frac{2n+1}{2}\right)^n.
\end{aligned}$$

Now the number of antipalindromes of length $2n$ is simply 2^n (one for each choice of the variables X_j). Therefore the sum of the costs of all antipalindromes of length $2n$ is simply 2^n multiplied by the expected value of the cost of P. This is

$$2^n \times \left(\frac{2n+1}{2}\right)^n = (2n+1)^n.$$