New Zealand Mathematical Olympiad Committee

2019 NZMO Round 2

September 2019

Instructions

1. You have 3 hours to work on the exam.
2. There are 5 problems, each worth equal marks. You should attempt all 5 problems. You may work on them in any order.
3. Geometrical instruments (ruler and compasses) may be used. Calculators, phones, computers and electronic devices of any sort are not permitted.
4. Write your solutions on the paper provided. Let the supervisor know if you need more. Write your name at the top of every page as you start using it.
5. Full written solutions — not just answers — are required, with complete proofs of any assertions you make. Your marks will depend on the clarity of your mathematical presentation. Work in rough first, and then draft a neat final version of your best attempt.
6. Make sure you fill in your details on the cover sheet. Hand in all of your rough work, in addition to your neat solutions.
7. At the end of the exam, remain seated quietly until all scripts have been collected and the supervisor indicates that you are free to move.
8. You may not take the question paper from the exam room.
9. The contest problems are to be kept confidential until they are posted on the NZMOC webpage `www.mathsolympiad.org.nz`. Do not disclose or discuss the problems online until this has occurred.
10. **Do not turn over until told to do so**.

Problems

1. A positive integer is called *sparkly* if it has exactly 9 digits, and for any n between 1 and 9 (inclusive), the n^{th} digit is a positive multiple of n. How many positive integers are sparkly?

2. Let X be the intersection of the diagonals AC and BD of convex quadrilateral $ABCD$. Let P be the intersection of lines AB and CD, and let Q be the intersection of lines PX and AD. Suppose that $\angle ABX = \angle XCD = 90°$. Prove that QP is the angle bisector of $\angle BQC$.

3. Let a, b and c be positive real numbers such that $a + b + c = 3$. Prove that
$$a^a + b^b + c^c \geq 3.$$

4. Show that for all positive integers k, there exists a positive integer n such that $n2^k - 7$ is a perfect square.

5. An equilateral triangle is partitioned into smaller equilateral triangular pieces. Prove that two of the pieces are the same size.