

Roots, Reals and Radicals

10. Find all polynomials P such that $P(0) = 1$ and $P(x^2 + 1) = P(x)^2 + 1$.
What if $P(0) = 2$?
11. If p is a polynomial and $p(x) = p(-x)$ for all x , show that $p(x) = q(x^2)$ for some polynomial q .
12. P is a polynomial of degree n . Find $P(n + 1)$ in each case.
- (a) $P(0) = 1$ and $P(k) = 0$ for all $k \in \{1, 2, 3, \dots, n\}$.
 - (b) $P(k) = \frac{k}{k+1}$ for all $k \in \{0, 1, 2, \dots, n\}$.
 - (c) $P(k) = \binom{n+1}{k}^{-1}$ for all $k \in \{0, 1, 2, \dots, n\}$.

13. If P, Q, R, S are polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x),$$

prove that $(x - 1)$ is a factor of $P(x)$.

14. Suppose $|ax^3 + bx^2 + cx + d| \leq 1$ for all $|x| \leq 1$. Prove that

$$|a| + |b| + |c| + |d| \leq 7.$$

15. Let r be the greatest positive root of $x^3 - 3x^2 + 1 = 0$. Show that $\lfloor r^{1988} \rfloor$ is a multiple of 17.
16. Find the size of the set of all reals x such that

$$\sum_{k=0}^{2012} \frac{k}{x - k} \geq \frac{5}{4}.$$

[Hint: The set you are looking for is a disjoint union of 2012 intervals.
Find the sum of the lengths of these intervals.]

17. What is the area of a square with all four vertices on the graph of $y = x^3 + ax^2 + bx + c$?

Comments, hints or solutions? e-mail: rosstherock@gmail.com

Senior Polynomials

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Integer Involvement

1. When is $P(x) = x^n + x^{n-1} + \dots + x^2 + x + 1$ irreducible over the integers?
2. Let P be a polynomial with integer coefficients (an integer polynomial) such that $P(2012) = P(2013) = 2013$. Prove that P has no integer roots.
3. Let a, b, c be integers, and let P be an integer polynomial. If $P(a) = b$, $P(b) = c$ and $P(c) = a$ show that $a = b = c$.
4. Does there exist an integer polynomial P such that $P(n)$ is prime for all positive integers n ?
5. Suppose P is an integer polynomial with $\deg(P) = n$. Prove that there are at most $n + 2$ integers k such that $|P(k)| \leq 1$.
6. Find an integer polynomial with $2^{\frac{1}{6}} + 2^{-\frac{1}{6}}$ as a root?
7. Find integer polynomials P and Q , such that

$$\frac{P(\sqrt{2} + \sqrt{3} + \sqrt{5})}{Q(\sqrt{2} + \sqrt{3} + \sqrt{5})} = \sqrt{2} + \sqrt{3}.$$

8. Ivan's favourite polynomial, P , has positive integer coefficients. Every morning, whenever Konrad arrives n minutes late for breakfast, Ivan will arrive $P(n)$ minutes later. How many breakfasts are required for Konrad to determine P ?