NZMO Round One 2024 — Solutions

1. **Problem:** Josie and Kevin are each thinking of a two digit positive integer. Josie's number is twice as big as Kevin's. One digit of Kevin's number is equal to the sum of digits of Josie's number. The other digit of Kevin's number is equal to the difference between the digits of Josie's number. What is the sum of Kevin and Josie's numbers?

   Solution: (James Xu)
   We'll use $\overline{AB}$ to denote a 2 digit number with A in the tens digit and B in the unit digit.
   Let Josie pick the number $\overline{AB}$ and Kevin $\overline{CD}$. Then we have

   $$\overline{AB} = 2 \times \overline{CD}$$

   $$10A + B = 20C + 2D$$

   Now, $A \geq 2C > C$ so $C \neq A + B$. Therefore, $C = |A - B|$ and $D = A + B$.

   - Case 1: $A \geq B$. So $C = |A - B| = A - B$. This yields $10A + B = 20C + 2D = 20(A - B) + 2(A + B)$. Which simplifies to give

     $$19B = 12A.$$

     This can only happen when A is a multiple of 19 which is impossible since $A > 0$ is a digit.

   - Case 2: $A < B$. $C = |A - B| = B - A$. This yields $10A + B = 20C + 2D = 20(B - A) + 2(A + B)$. Which simplifies to give

     $$4A = 3B.$$

     Since A and B are digits (and A is nonzero), this means either: $(A, B) = (3, 4)$ or $(A, B) = (6, 8)$.

   To verify: $\overline{AB} = 34 \Rightarrow \overline{CD} = 17$ which works,
   while $\overline{AB} = 68 \Rightarrow \overline{CD} = 34$ does not work.
   Therefore the final answer is $34 + 17 = 51$.

2. **Problem:** Prove the following inequality

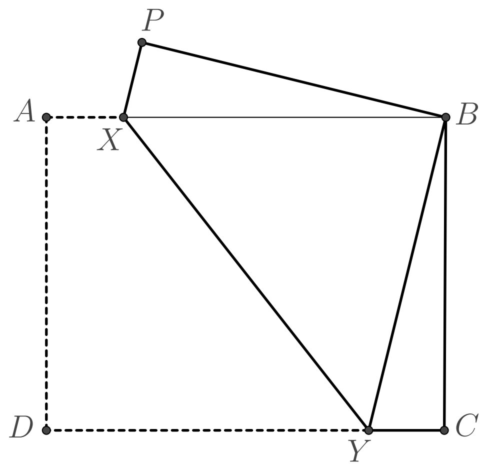
$$\frac{6}{2024^3} < \left(1-\frac{3}{4}\right)\left(1-\frac{3}{5}\right)\left(1-\frac{3}{6}\right)\left(1-\frac{3}{7}\right)\dots\left(1-\frac{3}{2025}\right).$$

Solution: (Eric Liang)

$$\begin{aligned}
&\left(1-\frac{3}{4}\right)\left(1-\frac{3}{5}\right)\left(1-\frac{3}{6}\right)\left(1-\frac{3}{7}\right)\dots\left(1-\frac{3}{2025}\right)\\
&=\frac{1}{4}\times\frac{2}{5}\times\frac{3}{6}\times\frac{4}{7}\times\frac{5}{8}\times\frac{6}{9}\times\frac{7}{10}\times\dots\times\frac{2022}{2025}\\
&=\frac{1\times2\times3\times4\times5\times6\times7\times\cdots\times2022}{4\times5\times6\times7\times8\times9\times10\times\dots\times2025}\\
&=\frac{1\times2\times3}{2023\times2024\times2025}\\
&=\frac{6}{2024(2024-1)(2024+1)}\\
&=\frac{6}{2024(2024^2-1)}\\
&>\frac{6}{2024(2024^2)}\\
&=\frac{6}{2024^3}.
\end{aligned}$$

3. **Problem:** A rectangular sheet of paper is folded so that one corner lies on top of the corner diagonally opposite. The resulting shape is a pentagon whose area is 20% one-sheet-thick, and 80% two-sheets-thick. Determine the ratio of the two sides of the original sheet of paper.

Solution: (Ross Atkins)
Let the original rectangle be $ABCD$. Let the fold line be XY with point X lying on side AB and point Y lying on side CD. Consider the fold such that point D lands on point B. Let P be the location of corner A after the fold.



The two-sheets-thick area is $\triangle BXY$ which should be 80%. By symmetry, triangles $\triangle PXB$ and $\triangle CYB$ are congruent with equal hypotenuses

$$BX = BY.$$

Triangles $\triangle PXB$ and $\triangle CYB$ have equal area and their total area is 20% of rectangle $ABCD$. Therefore $\triangle BYC$ has area 10%. Hence the ratio of the areas $|BYC| : |BXY|$ is 10% : 80%. Both triangles have height BC equal to the height of rectangle $ABCD$. Therefore their bases are also in the ratio $1 : 8$. *i.e.* $XB = 8YC$.

$$\therefore BY = XB = 8YC.$$

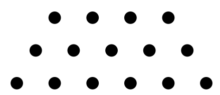
Now we apply Pythagoras' to $\triangle BCY$ to get

$$BC^2 = BY^2 - CY^2 = (8CY)^2 - CY^2 = 63CY^2.$$

Therefore $BC = \sqrt{63}CY$.
Combining this with $CD = CY + YD = CY + YB = 9CY$ we get

$$BC : CD = \sqrt{7} : 3.$$

4. **Problem:** A dot-trapezium consists of several rows of dots such that each row contains one more dot than the row immediately above (apart from the top row). For example here is a dot-trapezium consisting of 15 dots, having 3 rows and 4 dot in the top row.



A positive integer n is called a trapezium-number if there exists a dot-trapezium consisting of exactly n dots, with at least two rows and at least two dots in the top row. How many trapezium-numbers are there less than 100?

Solution A: (Chris Tuffley)
Let n be a trapezium number and suppose there are a dots in the first row and b dots in the last row. So the required conditions are $a \geq 2$ and $b \geqslant a+1$. Then the equation becomes:

$$2n = b(b+1) - a(a-1) = b^2 - a^2 + b + a = (a+b)(b-a+1)$$

because it is the difference between two triangle numbers. Let the two factors on the RHS be $x = (a+b)$ and $y = (b-a+1)$. Rearranging gives us

$$a = \frac{x-y+1}{2} \qquad \text{and} \qquad b = \frac{x+y-1}{2}.$$

In order for a and b to be integers we must have x and y being opposite parity. So we are looking for factorizations of the form $2n = xy$ such that one of x and y is even while the other is odd. We also need $a < b$ (so that the trapezium has at least two rows) which is equivalent to $y \geqslant 2$. Finally we also need $a \geq 2$ (so there are at least two dots in the top row) which is equivalent to $x \geqslant y+3$. We need

Now write $n = 2^k m$ with m odd.

- If $n = 2^k$ is a power of 2, then the only factorization of $2n = 2^{k+1} = xy$ such that both x and y have opposite parity is $x = 2^{k+1}$ and $y = 1$. This doesn't work because we require $y > 1$. (when $y = 1$ the "trapezium" would consist of only one row)
- If $m > 2^{k+1} + 1$ or $1 < m < 2^{k+1} - 1$ then we can choose

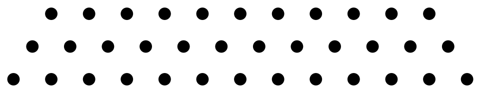
  $$x = \max\{2^{k+1}, m\} \quad \text{and} \quad y = \min\{2^{k+1}, m\}.$$

  To check that this works we simply check that $y > 1$ and $x \geqslant y + 3$. Note that this still works even when $k = 0$.
- If $m = 2^{k+1} \pm 1$ and m is prime then the only factorization of $2n = 2^{k+1}m = xy$ such that both x and y have opposite parity and $x > y > 1$ is

  $$x = \max\{2^{k+1}, m\} \quad \text{and} \quad y = \min\{2^{k+1}, m\}.$$

  However this doesn't work because we require $x > x + 1$. (when $y = 1$ the "trapezium" would consist have one dot in the top row)

The only remaining possibility is when $m = 2^{k+1} \pm 1$ and m is composite. If $k > 2$ then $m = 2^{k+1} \pm 1 \geq 15$, so $n = 2^k m \geq 8 \times 15 > 100$ and we don't need to consider it. For $k \leq 2$ we get $m = 1, 3, 5, 7, 9$. We can't have $m = 1$ because then n would be a power of 2. We also can't have $m = 3, 5, 7$ because they are prime. Finally we consider $m = 9$ and so $k = 2$. In this case we get $n = 2^k m = 36$, which is a trapezium number as seen here:



Therefore all non-trapezium-numbers less than 100, are the powers of two and numbers of the form $n = 2^k(2^{k+1} \pm 1)$ where $(2^{k+1} \pm 1)$ is prime (and $k \leq 2$). The powers of two are: $\{1, 2, 4, 8, 16, 32, 64\}$. The numbers of the form $2^k(2^{k+1} \pm 1)$ with $k \leq 2$ and $(2^{k+1} \pm 1)$ being prime are

$$\{2^0(2^1 + 1) = 3, 2^1(2^2 - 1) = 6, 2^1(2^2 + 1) = 10, 2^2(2^3 - 1) = 28\}.$$

All together the non-trapezium numbers are $\{1, 2, 3, 4, 6, 8, 10, 16, 28, 32, 64\}$. There are 99 positive integers less than 100 and exactly 11 of them are non-trapezium numbers. So the final answer is $99 - 11 = 88$.

Solution B: (David Starshaw)
First, notice that any odd integer can be written as a sum of consecutive integers: $2n + 1 = (n) + (n + 1)$, e.g. $23 = 11 + 12$.
Then observe that any multiple of 3 can be written as a sum of consecutive integers: $3n = (n - 1) + (n) + (n + 1)$, for example, $18 = 5 + 6 + 7$.
Similarly, any multiple of 5 can be written as a sum of consecutive integers:

$$5n = (n - 2) + (n - 1) + (n) + (n + 1) + (n + 2).$$

In general, any number with an odd factor can be written as a sum of consecutive integers. The only numbers that have no odd factors (other than 1) are the powers of two. There are 7 such numbers less than 100: $\{1, 2, 4, 8, 16, 32, 64\}$.
In general, any number with an odd factor (greater than one) can be written as a sum of consecutive integers. Note that if the odd factor is too large then some of these terms are negative. For example our above method for 26 would produce

$$26 = (-4) + (-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8.$$

Because the centre of the consecutive run is always positive, there will always be more positive terms than negative terms. Therefore each negative term can be 'cancelled' with it's corresponding positive term. At a minimum, the centre of the run must be at least 1, so there will always be at least two positive terms remaining even after cancelling.
But we are also not allowed to have one dot in the top row, i.e. our sum (after cancelling) cannot include 1. This would seem to exclude the triangular numbers:

$$\{1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91\}.$$

Except some can be rewritten as a different sum of consecutive integers in more than one way. For example $15 = 1 + 2 + 3 + 4 + 5$, but also $15 = 7 + 8$. In general, if the triangular number, $T_n = \frac{1}{2}n(n + 1)$, is either odd, or has an odd factor less than n, it can be rewritten as a sum of consecutive integers that do not include 1.
The following triangular numbers can be salvaged:

- 15 is odd, so $15 = 7 + 8$
- 21 is odd, so $21 = 10 + 11$
- 36 is a multiple of 3, so $36 = 11 + 12 + 13$
- 45 is odd, so $45 = 22 + 23$
- 55 is odd, so $55 = 27 + 28$
- 66 is a multiple of 3, so $66 = 21 + 22 + 23$
- 78 is a multiple of 3, so $78 = 25 + 26 + 27$
- 91 is odd, so $91 = 45 + 46$

So there are 5 triangular numbers that are not possible: $\{1, 3, 6, 10, 28\}$. Combined with the powers of two that are also impossible: $\{1, 2, 4, 8, 16, 32, 64\}$, there are 11 numbers ($5 + 7 = 12$ but the number 1 is included in both lists) that can not be written as trapezium numbers:

$$\{1, 2, 3, 4, 6, 8, 10, 16, 28, 32, 64\}$$

There are 99 numbers less than 100, so there are $99 - 11 = 88$ trapezium numbers.

5. **Problem:** A shop sells golf balls, golf clubs and golf hats. Golf balls can be purchased at a rate of 25 cents for two balls. Golf hats cost \$1 each. Golf clubs cost \$10 each. At this shop, Ross purchased 100 items for a total cost of exactly \$100 (Ross purchased at least one of each type of item). How many golf hats did Ross purchase?

   Solution: (James Xu)
   Let the number of pairs of balls, clubs, and hats purchased be b, c, h respectively. Then, we must have

$$\text{cost:} \qquad \frac{1}{4}b + h + 10c = 100$$
$$\text{quantity:} \qquad 2b + h + c = 100$$

   Subtracting these yields $\frac{7}{4}b - 9c = 0$ and thus

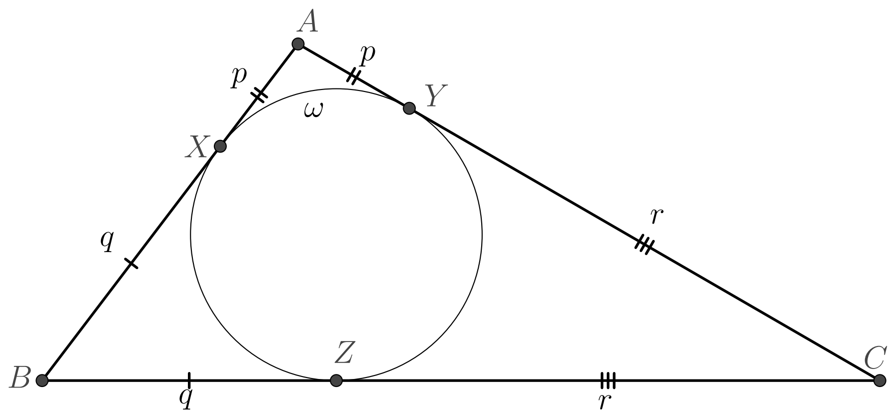
$$7b = 36c.$$

   Since $\gcd(7, 36) = 1$ this implies $36|b$. Since $0 < b < 51$, we must have $b = 36$. Therefore $c = 7$. Thus $h = 100 - 2 \times 36 - 7 = 21$. Hence the final answer is $(b, h, c) = (36, 21, 7)$. Checking we get:

$$\text{cost:} \qquad \frac{1}{4}b + h + 10c = 9 + 21 + 70 = 100$$
$$\text{quantity:} \qquad 2b + h + c = 72 + 21 + 7 = 100.$$

6. **Problem:** Let ω be the incircle of scalene triangle ABC. Let ω be tangent to AB and AC at points X and Y. Construct points X' and Y' on line segments AB and AC respectively such that $AX' = XB$ and $AY' = YC$. Let line CX' intersects ω at points P, Q such that P is closer to C than Q. Also let R be the intersection of lines CX' and BY'. Prove that $CP = RX'$.

   Solution: (Ross Atkins)
   Let a, b, c be the sidelengths BC, AC, AB respectively, and let s be the semiperimeter of triangle ABC (*i.e.* let $s = \frac{a+b+c}{2}$). Since X and Y are the points of contact of the incircle we get $AX = AY$. Similarly $BX = BZ$ and $CY = CZ$ where Z is the point of tangency between ω and side BC. Let $p = AX = AY$ and $q = BX = BZ$ and $r = CY + CZ$ as in the diagram.



   We have the system of equations:

$$p + q = c$$
$$p + r = b$$
$$q + r = a$$

   Adding them together yields $2(p + q + r) = a + b + c$ so therefore $p + q + r = s$. Then we simply get:

$$s - c = (p + q + r) - (p + q) = r.$$

   Similarly $p = (s - a)$ and $q = (s - b)$. Hence $BX' = AX = AY = CY' = (s - a)$, and $AX' = (s - b)$.

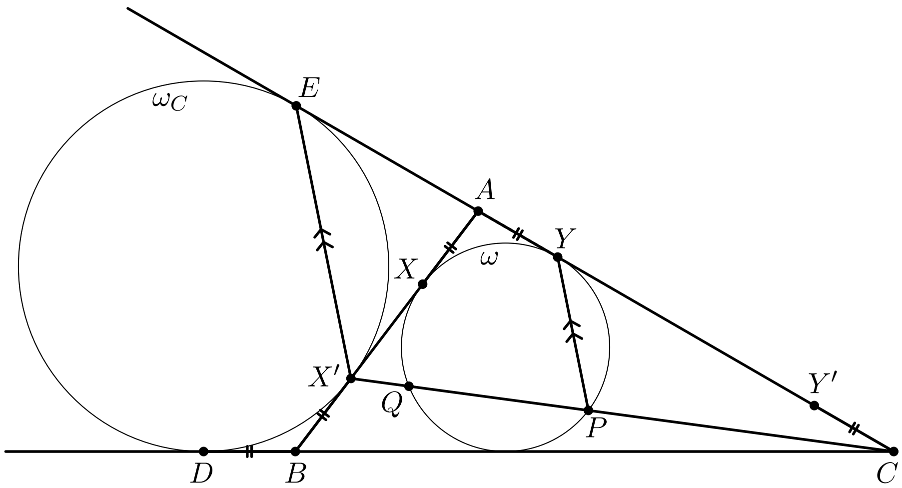
Now let ω_C be the excircle of triangle ABC opposite vertex C. The circle ω_C is tangent to lines BC, AC and AB at points D, E and F respectively. First we consider equal tangents from C to ω_C.

$$CD = BC + BD = a + BF$$
$$CE = CA + AE = b + AF$$

Adding these together gives us $CD + CE = a + b + (AF + BF) = a + b + c$. But since $CD = CE$ (equal tangents) this implies

$$CD = \frac{a+b+c}{2} = s.$$

Therefore $CD = CE = s$. This means $BF = CD - a = (s - a)$ and thus X' and F are the same point.



Now consider the homothety (centred at C) that carries ω to ω_C. This homothety sends point P to point X' (since C, P, X' are colinear). It also carries the tangency point Y to E. It follows that

$$\frac{CP}{PX'} = \frac{CY}{YE} = \frac{CY}{CE - CY} = \frac{s-c}{s-(s-c)} = \frac{s-c}{c}.$$

i.e. Point P divides segment CX' into the ratio $(s - c) : c$. Also we can apply Menelaus' Theorem (B, R, Y' colinear) to get

$$\frac{CR}{RX'} \times \frac{X'B}{BA} \times \frac{AY'}{Y'C} = 1$$
$$\frac{CR}{RX'} \times \frac{s-a}{c} \times \frac{s-c}{s-a} = 1$$

Hence $\dfrac{CR}{RX'} = \dfrac{c}{s-c}$, so point R divides segment CX' into the ratio $c : (s - c)$.

Therefore $CP = X'R$ as required.

7. **Problem:** Some of the 80960 lattice points in a 40×2024 lattice are coloured red. It is known that no four red lattice points are vertices of a rectangle with sides parallel to the axes of the lattice. What is the maximum possible number of red points in the lattice?

   Solution: (Ross Atkins)
   Let $a_1, a_2, a_3, \ldots, a_{2024}$ be the number of red dots in rows $1, 2, 3, \ldots, 2024$ respectively. So $0 \leqslant a_i \leqslant 40$ for each i.
   For each of the $\binom{40}{2} = 780$ pairs of columns, there can be at most one row with a red dot in both columns. Therefore we must have

$$\binom{a_1}{2} + \binom{a_2}{2} + \binom{a_3}{2} + \cdots + \binom{a_{2024}}{2} \leq \binom{40}{2} = 780.$$

   And hence there is always guaranteed to be at least $2024 - 780$ indices with $a_i \leqslant 1$ - *i.e.* At least $2024 - 780 = 1244$ rows have at most one red dot in it.

   Now consider an arrangement in which the total number of red dots is maximised, and suppose for the sake of contradiction that $a_j \geqslant 3$ for some index j. Let c_1, c_2 and c_3 be three of the columns where row j has a red dot. Let r_1 and r_2 be any two rows which each currently contain one red dot. Consider the following operation:

   - Remove the dot in row j and column c_3.
   - Remove the dots in rows r_1 and r_2 (at most two dots removed).
   - Add dots in columns c_1 and c_3 in row r_1.
   - Add dots in columns c_2 and c_3 in row r_2.

   This operation will increase the total number of dots which contradicts the assumption that a row with at least 3 dots exists in a maximal arrangement.

   Henceforth we assume $a_i \leqslant 2$ for all i. Now if there were more than 780 rows with two red dots in it (*i.e.* $a_i = 2$ for more than 780 indicies i). Then by the pigeonhole principle, there would be two rows with dots in the same pair of columns and this would be a contradiction. Therefore there are at most 780 rows with two dots in it. Hence the total number of dots is at most

$$780 \times 2 + 1244 \times 1 = 2804.$$

   To achieve this let the first 780 rows each have a unique pair of columns in which their red dots lie. And the remaining 1244 rows each containing a single dot in the first column only.

8. **Problem:** Let a, b and c be any positive real numbers. Prove that

$$\frac{a^2+b^2}{2c}+\frac{a^2+c^2}{2b}+\frac{b^2+c^2}{2a} \geqslant a+b+c.$$

Solution A: (Eric Liang)
Since square numbers cannot be negative we trivially get

$$\left(\frac{y}{\sqrt{x}}-\sqrt{x}\right)^2 \geqslant 0$$

for any positive x and y. Expanding the brackets and rearranging gives us $\frac{y^2}{x}+x \geq 2y$. Since this holds for any x and y we get the following inequalities:

$$\begin{aligned}
\frac{a^2}{b}+b &\geq 2a\\
\frac{a^2}{c}+c &\geq 2a\\
\frac{b^2}{a}+a &\geq 2b\\
\frac{b^2}{c}+c &\geq 2b\\
\frac{c^2}{b}+b &\geq 2c\\
\frac{c^2}{a}+a &\geq 2c
\end{aligned}$$

Summing these together gives us

$$\frac{a^2+b^2}{c}+\frac{a^2+c^2}{b}+\frac{b^2+c^2}{a} \geqslant 2a+2b+2c$$

and then dividing by 2 gives us the required result.

Solution B: (Ross Atkins)
Since a, b are positive then $(a+b)/ab > 0$. Furthermore, since square numbers are nonegative, we must have $(a+b)(a-b)^2 \geq 0$. Expanding the brackets gives us,

$$(a+b)(a-b)^2 = a^3 - a^2b - ab^2 + b^3 \geq 0$$

Thus $a^3 + b^3 \geq a^2b + ab^2$. Dividing both sides by ab gives us

$$\frac{a^2}{b} + \frac{b^2}{a} \geq a + b.$$

Similarly we have $\frac{a^2}{c} + \frac{c^2}{a} \geq a + c$ and $\frac{b^2}{c} + \frac{c^2}{b} \geq b + c$. Adding these all together gives us

$$\left(\frac{a^2}{b} + \frac{b^2}{a}\right) + \left(\frac{a^2}{c} + \frac{c^2}{a}\right) + \left(\frac{b^2}{c} + \frac{c^2}{b}\right) \geq (a+b) + (a+c) + (b+c)$$

$$\implies \frac{a^2+b^2}{c} + \frac{a^2+c^2}{b} + \frac{b^2+c^2}{a} \geq 2(a+b+c)$$

$$\implies \frac{a^2+b^2}{2c} + \frac{a^2+c^2}{2b} + \frac{b^2+c^2}{2a} \geq a+b+c.$$

Solution C: (Eric Liang)
By the AM-GM (Arithmetic Mean Geometric Mean) Inequality we have $x^2 + y^2 \geq 2xy$ therefore

$$\frac{a^2+b^2}{2c} + \frac{a^2+c^2}{2b} + \frac{b^2+c^2}{2a} \geqslant \frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a}$$

Now without loss of generality assume $a \geq b \geq c$. So the sequences (ab, ac, bc) and $(\frac{1}{c}, \frac{1}{b}, \frac{1}{a})$ are similarly ordered. Hence we can apply the Rearrangement Inequality to get

$$\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} \geqslant a+b+c.$$

QED