New Zealand Mathematical Olympiad Committee

NZMO Round One 2023 — Solutions

1. **Problem:** There are 2023 employees in the office, each of them knowing exactly 1686 of the others. For any pair of employees they either both know each other or both don't know each other. Prove that we can find 7 employees each of them knowing all 6 others.

   Solution A: (Ross Atkins)
   If every person knows 1686 others then for each person, there are $2023 - 1686 - 1 = 336$ people that they don't know. Now consider any group of p people from the office. There will be at most $336p$ people who don't know someone in the group (336 for each person in the group). Therefore there are at least $(2023 - p) - 336p$ people not in the group who know everyone in the group. If $p \leq 6$ then

   $$(2023 - p) - 336p \geq (2023 - 6) - 336 \times 6 = 1.$$

   So for any group of at most 6 people, there exists at least one person not in the group that knows everyone in the group. Hence we perform the following process.

   - Start with a random group of $p = 2$ people who know each other, and then
   - while $p \leq 6$ choose a person who knows all the current members of the group (at random) and add them to the group.

   This process ends with a group of 7 people each knowing everyone in the group.

   Solution B: (Ishan Nath)
   We will show by induction that, for non-negative integers k and $n > 0$, if there are $nk + 1$ people such that any of them know at least $n(k-1) + 1$ of the others, then there are $k + 1$ people who all know each other. For $k = 0$ this is true, as we have one person.

   Now assume this is true for some integer k. Among any $n(k+1) + 1$ people who all know at least $nk + 1$ of the others, pick an arbitrary person P, and a set $\mathcal{S}$ of $nk + 1$ people that P knows.

   For any person in $\mathcal{S}$, they must know at least $n(k-1)+1$ others in $\mathcal{S}$, as there are exactly n people outside of $\mathcal{S}$, and they know at least $nk + 1$ people in total. Hence by our induction hypothesis, $\mathcal{S}$ contains $k + 1$ people who all know each other.

   As P knows everyone in $\mathcal{S}$, including P gives a group of $k + 2$ people who all know each other, proving our inductive result.

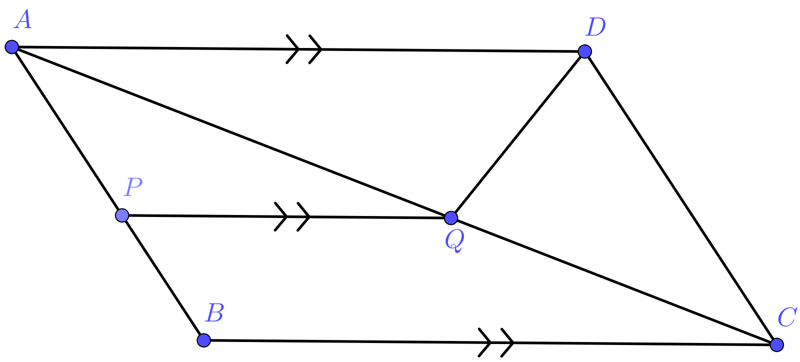
   Now letting $n = 337$ and $k = 6$, we get the desired result.

2. **Problem:** Let $ABCD$ be a parallelogram, and let P be a point on the side AB. Let the line through P parallel to BC intersect the diagonal AC at point Q. Prove that

$$|DAQ|^2 = |PAQ| \times |BCD|,$$

where $|XYZ|$ denotes the area of triangle XYZ.

Solution: (Kevin Shen)



Since PQ is parallel to BC, there is a dilation (centred at A) of factor AB/AP that sends triangle APQ to triangle ABC. Thus

$$|PAQ| = |ABC| \times \left(\frac{AP}{AB}\right)^2.$$

Since AD is parallel to BC, triangle ABC has the same height as triangle BCD. Triangles ABC and BCD also have a common base, so $|ABC| = |BCD|$. Similarly $|DAQ| = |DAP|$ (because $PQ||AD$ and AD is a common base).

$$\begin{aligned}
\therefore |DAP| &= |DAB| \times \frac{AP}{AB} \\
&= |BCD| \times \frac{AP}{AB} && \text{(diagonal splits parallelograms in half)} \\
|DAP|^2 &= |BCD| \times |BCD| \times \left(\frac{AP}{AB}\right)^2 \\
&= |BCD| \times |PAQ| && \text{(because } |PAQ| = |ABC|\dot{(}AP/AB)^2\text{)}
\end{aligned}$$

as required.

3. **Problem:** Find the sum of the smallest and largest possible values for x which satisfy the following equation.

$$9^{x+1} + 2187 = 3^{6x-x^2}$$

Solution: (Viet Hoang)
First we prove that the given equation has at least one root. To do this we consider the following function.

$$f(x) = 9^{x+1} + 2187 - 3^{6x-x^2}$$

Note that this function is continuous. Note also that $f(0) = 9 + 2197 - 1 > 0$ and $f(3) = 9^4 + 2187 - 3^9 < 0$. Therefore by the *intermediate value theorem* there exists a real number $0 < z < 3$ such that $f(z) = 0$. Therefore there is at least one value x which statisfies the equation.

Next, we can rewrite the equation in the following form

$$3^x + 3^{5-x} = 3^{x(5-x)-2} \tag{1}$$

Let r and s be any two numbers such that $r + s = 5$. Notice that if r is a root of 1 then s must also be a root (and vice versa). This is because

$$3^r + 3^{5-r} = 3^{5-s} + 3^s \qquad \text{and} \qquad 3^{r(5-r)-2} = 3^{(5-s)s-2}.$$

Now we claim that if x_0 is the smallest root of 1, then $(5 - x_0)$ must be the largest root. Indeed if x_1 were a root of 1 greater than $(5 - x_0)$, then $(5 - x_1)$ would also have to be a root, but $(5 - x_1) < (5 - (5 - x_0)) = x_0$ and this would contradict the fact that x_0 is the smallest root.
Therefore, the sum of the largest and the smallest roots of 1 is $x_0 + (5 - x_0) = 5$.

4. **Problem:** Let p be a prime and let $f(x) = ax^2 + bx + c$ be a quadratic polynomial with integer coefficients such that $0 < a, b, c \leqslant p$. Suppose $f(x)$ is divisible by p whenever x is a positive integer. Find all possible values of $a + b + c$.

   Solution: (James Xu)
   First substitute $x = p$, to get $p \mid ap^2 + bp + c$ so $p \mid c$. Therefore $c = p$. Next substitute $x = 1$, to get $p \mid a + b + c$. Since $c = p$, this gives us $p \mid a + b$. Finally substitute $x = p - 1$, to get $(p-1)^2 a - (p-1)b + c \equiv 0 \pmod{p}$. Thus $a - b + c \equiv 0 \pmod{p}$, and since $c = p$, we get $p \mid a - b$. Adding and subtracting $p \mid a + b$ and $p \mid a - b$ we get

   $$p \mid (a+b) + (a-b) = 2a \qquad \text{and} \qquad p \mid (a+b) - (a-b) = 2b.$$

   Now there are two cases: either $p = 2$ or p is odd.

   - Case 1 $p = 2$:
     $2 \mid a - b$ means that either $a = b = 1$ or $a = b = 2$, *i.e.* the only two possibilities for (a, b, c) are $(1, 1, 2)$ and $(2, 2, 2)$. So the only candidate polynomials are

     $$x^2 + x + 2 \qquad \text{and} \qquad 2x^2 + 2x + 2.$$

     Now we check that these work. If $f(x) = x^2 + x + 2 = 2 + x(x+1)$ which is always even because both 2 and $x(x+1)$ are even for all integers x. If $f(x) = 2x^2 + 2x + 2 = 2(x^2 + x + 1)$ which is always even because it has a factor of 2.

   - Case 2 p is an odd prime:
     Since $(2, p) = 1$, $p \mid a$ so $a = p$. Also note that $p \mid 2b$ by subtracting $a - b$ from $a + b$, so by a similar argument, $b = p$. Thus, $a + b + c = p + p + p = 3p$

   Combining the two cases, $a + b + c = 3p$ for all p and $a + b + c = 4$ when $p = 2$ are the only possible values of $a + b + c$.

5. **Problem:** Find all triples (a, b, n) of positive integers such that a and b are both divisors of n, and $a + b = \frac{n}{2}$.

Solution: (James Xu)
Since a and b are both factors of n, we can find positive integers x and y such that $a = \frac{n}{x}$ and $b = \frac{n}{y}$. Then $\frac{n}{x} + \frac{n}{y} = \frac{n}{2}$ so

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2}.$$

Without loss of generality assume $a \leqslant b$. So $x \geqslant y$ and $\frac{1}{x} \leqslant \frac{1}{y}$ and thus

$$\frac{2}{y} \geqslant \frac{1}{x} + \frac{1}{y} = \frac{1}{2}.$$

Therefore $y \leqslant 4$. So we try $y = 1, 2, 3, 4$ one-by-one.

- Case 1 $y = 1$:
  This implies $\frac{1}{x} = \frac{1}{2} - \frac{1}{y} = 0$ thus $x = -2$. This doesn't work because a (and therefore x) must be positive.
- Case 2 $y = 2$:
  This implies $\frac{1}{x} = \frac{1}{2} - \frac{1}{y} = 0$ which is not possible.
- Case 3 $y = 3$:
  This implies $\frac{1}{x} = \frac{1}{2} - \frac{1}{y} = \frac{1}{6}$ and thus $x = 6$. This means that n must be a multiple of 6. Now let $n = 6k$, and we get

  $$a = \frac{n}{x} = \frac{6k}{6} = k \qquad \text{and} \qquad b = \frac{n}{y} = \frac{6k}{3} = 2k.$$

  This yields the family of solutions $(a, b, n) = (k, 2k, 6k)$ where k is any positive integer.
- Case 4 $y = 4$:
  This implies $\frac{1}{x} = \frac{1}{2} - \frac{1}{y} = \frac{1}{4}$ and thus $x = 4$. This means that n must be a multiple of 4. Now let $n = 4k$, and we get

  $$a = \frac{n}{x} = \frac{4k}{4} = k \qquad \text{and} \qquad b = \frac{n}{y} = \frac{4k}{4} = k.$$

  This yields the family of solutions $(a, b, n) = (k, k, 4k)$ where k is any positive integer.

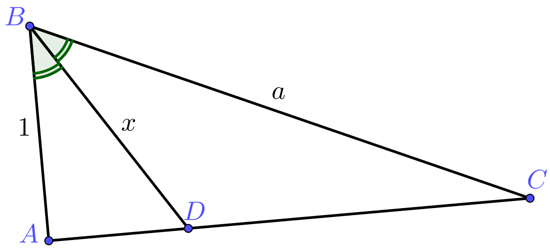
Remember that we assumed $a \leqslant b$ so we still need to swap the roles of a and b for our final answer. So in summary, the triples (a, b, n) which satisfy the given equation are

$$(k, 2k, 6k), (2k, k, 6k) \text{ and } (k, k, 4k)$$

where k is any positive integer.

6. **Problem:** Let triangle ABC be right-angled at A. Let D be the point on AC such that BD bisects angle $\angle ABC$. Prove that $BC - BD = 2AB$ if and only if $\frac{1}{BD} - \frac{1}{BC} = \frac{1}{2AB}$.

Solution: (Kevin Shen)
Wlog let $AB = 1$ and $BC = a$. Also let $BD = x$. We will try to find all the lengths in the diagram in terms of a.



By Pythagoras in $\triangle ABC$ we get $AC = \sqrt{a^2-1}$. By the angle-bisector theorem we get $\frac{AD}{DC} = \frac{AB}{BC} = \frac{1}{a}$, and so $DC = a \times AD$. This can be substituted into $AD + DC = AC = \sqrt{a^2-1}$ to get $AD(1+a) = \sqrt{a^2-1}$. Therefore

$$AD = \frac{AC}{1+a} = \frac{\sqrt{a^2-1}}{a+1}.$$

Hence $AD^2 = \frac{a^2-1}{(a+1)^2} = \frac{a-1}{a+1}$. Now consider Pythagoras in triangle $\triangle BAD$.

$$x^2 = 1^2 + AD^2 = 1 + \frac{a-1}{a+1} = \frac{2a}{a+1}.$$

Now we do the two directions of the 'if and only if' separately:

- First, assuming $\frac{1}{x} - \frac{1}{a} = \frac{1}{2}$ we get $x = \frac{2a}{a+2}$. Hence

$$\frac{2a}{a+1} = x^2 = \left(\frac{2a}{a+2}\right)^2 = \frac{4a^2}{a^2+4a+4}$$
$$a^2 + 4a + 4 = 2a(a+1)$$
$$0 = a^2 - 2a - 4$$

  So by the quadratic formula we get $a = 1 \pm \sqrt{5}$ but since $a > 0$ we must have $a = 1 + \sqrt{5}$. Therefore $x = \frac{2a}{a+2} = \frac{2(1+\sqrt{5})}{(1+\sqrt{5})+2} = \sqrt{5} - 1$.

$$\therefore a - x = (1+\sqrt{5}) - (\sqrt{5}-1) = 2.$$

  So $\frac{1}{x} - \frac{1}{a} = \frac{1}{2}$ implies $a - x = 2$.

- Second, assuming $a - x = 2$ we get $x = a - 2$. Hence

$$\frac{2a}{a+1} = x^2 = (a-2)^2 = a^2 - 4a + 4$$
$$2a = (a^2 - 4a + 4)(a+1)$$
$$0 = a^3 - 3a^2 - 2a + 4$$
$$0 = (a-1)(a^2 - 2a - 4)$$

  Since $a - x = 2$ we have $a > 2$ so $(a-1) \neq 0$ and thus $a^2 - 2a - 4 = 0$. This gives us $a = 1 + \sqrt{5}$ and so again $x = a - 2 = \sqrt{5} - 1$.

$$\therefore \frac{1}{x} - \frac{1}{a} = \frac{1}{\sqrt{5}-1} - \frac{1}{\sqrt{5}+1} = \frac{1}{2}.$$

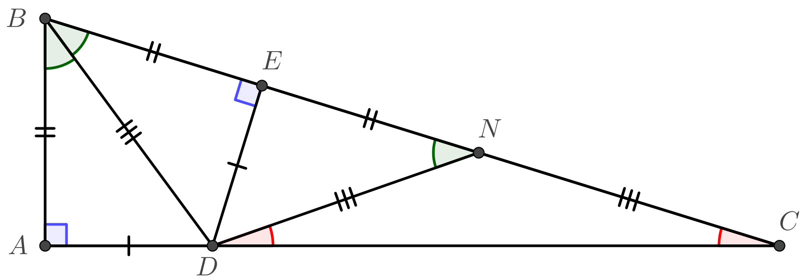
  So $a - x = 2$ implies $\frac{1}{x} - \frac{1}{a} = \frac{1}{2}$.

In summary $\frac{1}{x} - \frac{1}{a} = \frac{1}{2}$ is equivalent to $a - x = 2$.

Solution B: (Celine Yuan)
Let E be the foot of the perpendicular from D to BC. Note that E is the reflection of A about BD (because BD is an angle bisector) and so we have congruent triangles $\triangle ABD \equiv \triangle EBD$. Also construct point N on side BC to be the reflection of B about line DE. So we have three congruent triangles:

$$\triangle ABD \equiv \triangle EBD \equiv \triangle END.$$

Therefore $AB = EB = EN$ and hence $2AB = BE + EN = BN$. We also get $BD = DN$. Let F be the point on line BC such $BD = BF$.



- Assume $BC - BD = 2AB$. Since $BD = DN$ and $2AB = BN$ we can get $BC - DN = BN$. Therefore

$$DN = BC - BN = CN$$

and so triangle DNC is isosceles. Now let $\theta = \angle BCA$. Since $\triangle DNC$ is isosceles, we get $\angle CDN = \theta$ and thus $\angle END = 2\theta$. Therefore $\angle EBD = 2\theta$ and $\angle DBA = 2\theta$ and thus $\angle CBA = 4\theta$. Thus

$$90^\circ = \angle BCA + \angle CBA = \theta + 4\theta.$$

Hence $\theta = 18^\circ$. Therefore

$$\frac{AB}{BD} = \cos(\angle ABD) = \cos(2\theta) = \cos(36^\circ) = \frac{1+\sqrt{5}}{4}$$

$$\text{and} \quad \frac{BC}{BD} = \frac{\sin(\angle CDB)}{\sin(\angle BCD)} = \frac{\sin(126^\circ)}{\sin(\angle 18^\circ)} = \frac{\sqrt{5}+1}{\sqrt{5}-1}$$

Therefore $1 - \frac{BD}{BC} = 1 - \frac{\sqrt{5}-1}{\sqrt{5}+1} = \frac{2}{\sqrt{5}+1} = \frac{BD}{2AB}$. Hence

$$\frac{1}{BD} - \frac{1}{BC} = \frac{1}{2AB}$$

as required.

- If $BC - BD > 2AB$ then using similar logic we can get $\angle BCA < 18^\circ$ which implies

$$\frac{1}{BD} - \frac{1}{BC} < \frac{1}{2AB}$$

- Conversely if $BC - BD < 2AB$ then we get $\angle BCA > 18^\circ$ which implies

$$\frac{1}{BD} - \frac{1}{BC} > \frac{1}{2AB}$$

Therefore we conclude that $BC - BD > 2AB$ if and only if $\frac{1}{BD} - \frac{1}{BC} < \frac{1}{2AB}$.

7. **Problem:** Let n, m be positive integers. Let $A_1, A_2, A_3, \ldots, A_m$ be sets such that $A_i \subseteq \{1, 2, 3, \ldots, n\}$ and $|A_i| = 3$ for all i (*i.e.* A_i consists of three different positive integers each at most n). Suppose for all $i < j$ we have

$$|A_i \cap A_j| \leqslant 1$$

(*i.e.* A_i and A_j have at most one element in common).

(a) Prove that $m \leqslant \frac{n(n-1)}{6}$.

(b) Show that for all $n \geq 3$ it is possible to have $m \geqslant \frac{(n-1)(n-2)}{6}$.

Solution: (Ishan Nath)
Each set A_i has exactly three pairs of elements. But each unordered pair chosen from $\{1, 2, \ldots, n\}$ can be in at most one such set. Therefore

$$\binom{n}{2} \geq 3m.$$

This establishes the required upper bound on the size of m.

Let T be the set of all triples, i.e.

$$T = \{(a, b, c) \mid 1 \leq a < b < c \leq n\}.$$

We now partition T into n parts, $T = T_0 \cup T_1 \cup T_2 \cup \cdots \cup T_{n-1}$ based on the residue of $a + b + c$ modulo n, i.e.

$$T_i = \{(a, b, c) \mid (a, b, c) \in T \text{ and } a + b + c \equiv i \pmod{n}\}$$

So we have $\binom{n}{3}$ triples (our pigeons) and n pigeonholes (the parts $T_0, T_1, T_2, \ldots, T_{n-1}$) therefore by the pigeonhole principle at least one part T_k must have at least

$$\frac{1}{n} \times \binom{n}{3}$$

triples in it. Now it suffices to show that T_k satisfies the problem. For the sake of contradiction, assume there exists two triples $(a, b, c), (a', b', c') \in T_k$ such that $a = a'$ and $b = b'$ but $c \neq c'$. This would imply

$$a + b + c \equiv j \equiv a' + b' + c' \pmod{n}.$$

Hence $c \equiv c' \pmod{n}$ and thus $c = c'$ and so we get $(a, b, c) = (a', b', c')$, contradiction. Therefore no such pair $(a, b, c), (a', b', c') \in T_k$ exists. Therefore, letting the elements of T_k be our sets A_i, we achieve

$$m \geq \frac{1}{n} \times \binom{n}{3} = \frac{(n-1)(n-2)}{6}$$

as required.

8. **Problem:** Find all non-zero real numbers a, b, c such that the following polynomial has four (not necessarily distinct) positive real roots.

$$P(x) = ax^4 - 8ax^3 + bx^2 - 32cx + 16c$$

Solution: (Viet Hoang)
Assume that $P(x)$ has 4 positive real roots x_1, x_2, x_3 and x_4. Using Viete's theorem, one can obtain the following equations

$$x_1 + x_2 + x_3 + x_4 = \frac{8a}{a} = 8 \tag{2}$$

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = \frac{b}{a} \tag{3}$$

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = \frac{32c}{a} \tag{4}$$

$$x_1x_2x_3x_4 = \frac{16c}{a} \tag{5}$$

From 4 and 5, we have

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = \frac{x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4}{x_1x_2x_3x_4i} = \frac{\frac{32c}{a}}{\frac{16c}{a}} = 2 \tag{6}$$

Using AM-GM (arithmetic mean geometric mean inequality) on the sums in equation (2) and (6) we get

$$\frac{8}{4} = \frac{x_1 + x_2 + x_3 + x_4}{4} \geqslant \sqrt[4]{x_1 \cdot x_2 \cdot x_3 \cdot x_4}$$
$$\frac{2}{4} = \frac{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}}{4} \geqslant \sqrt[4]{\frac{1}{x_1} \cdot \frac{1}{x_2} \cdot \frac{1}{x_3} \cdot \frac{1}{x_4}}$$

Multiplying these together yields $1 \geqslant 1$ and so equality must occur in both AM-GM's. Therefore $x_1 = x_2 = x_3 = x_4$. Furthermore for Equation (2) we get $x_1 = x_2 = x_3 = x_4 = 2$. Substituting this into equation (3) gives us $b = 24a$. Substituting this into equation (5) gives us $c = a$. So $(a, b, c) = (a, 24a, a)$ where a is any non-zero real number, and the polynomial $P(x)$ is

$$P(x) = a(x - x_1)(x - x_2)(x - x_3)(x - x_4) = a(x - 2)^4.$$