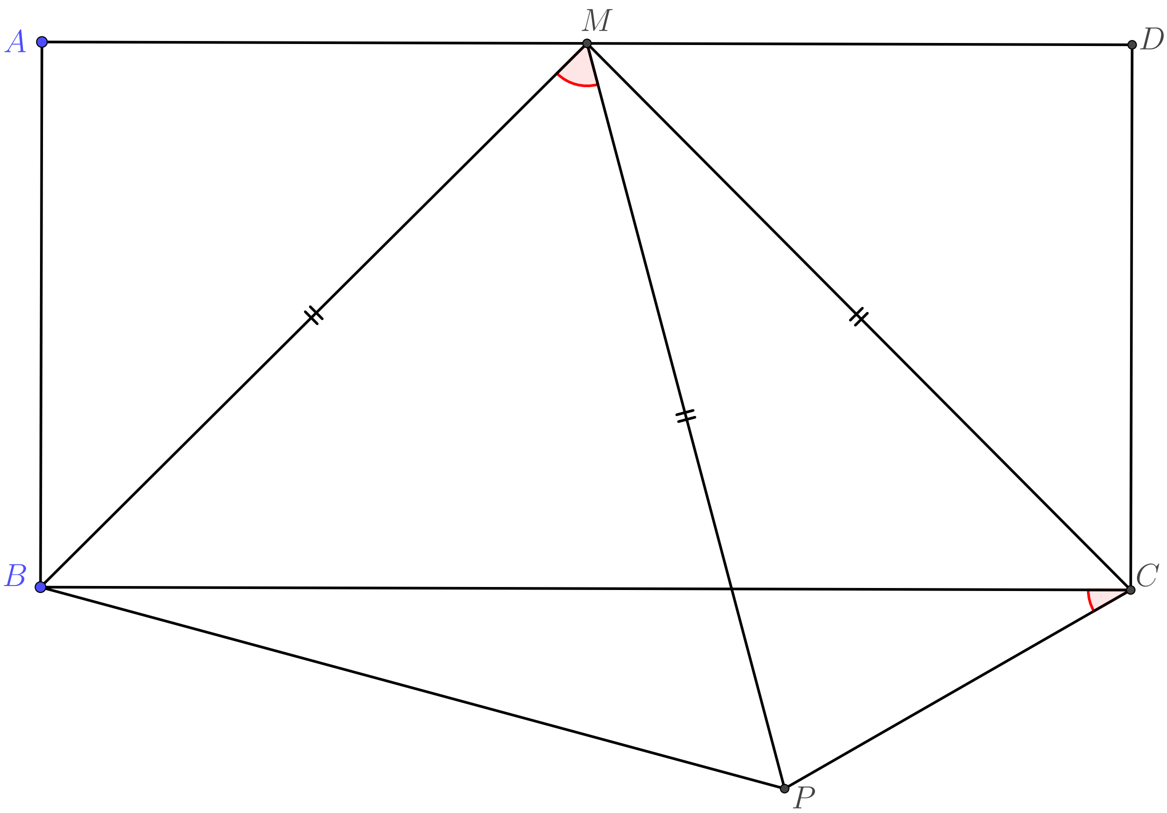
New Zealand Mathematical Olympiad Committee

NZMO Round One 2022 — Solutions

1. **Problem:** $ABCD$ is a rectangle with side lengths $AB = CD = 1$ and $BC = DA = 2$. Let M be the midpoint of AD. Point P lies on the opposite side of line MB to A, such that triangle MBP is equilateral. Find the value of $\angle PCB$.

   Solution: (Kevin Shen)
   M is the midpoint of AD, by symmetry $MB = MC$. The side lengths of an equilateral triangle are all equal, so $MB = MP$.



   As $MB = MC = MP$, M is the circumcenter of triangle BCP. For any chord of any circle, the angle subtended at the center is always double the angle subtended at the circumference. Since $\angle PCB$ and $\angle PMB$ are both subtended by arc BP, we get

$$\angle PCB = \frac{1}{2}\angle PMB = \frac{1}{2}60^\circ = 30^\circ$$

   as required.

2. **Problem:** Is it possible to pair up the numbers $0, 1, 2, 3, \ldots, 61$ in such a way that when we sum each pair, the product of the 31 numbers we get is a perfect fifth power?

Solution: (Jamie Craik)
Claim: It can be achieved. One way to achieve it is:

- to pair 0 with 1, and
- to pair k with $(63 - k)$ for $k = 2, 3, 4, \ldots, 31$.

This would result in a product equal to:

$$(0+1)\prod_{k=2}^{31}\big(k + (63-k)\big) = 1 \times 63^{30} = \big(63^6\big)^5$$

which is a perfect 5^{th} power of 63^6.

Note that there are many ways to solve this problem, for example:

$$\begin{aligned}(0+1) &\times (2+11)(3+10)(4+9)(5+8)(6+7)\\ &\times (12+21)(13+20)(14+19)(15+18)(16+17)\\ &\times (22+31)(23+30)(24+29)(25+28)(26+27)\\ &\times (32+41)(33+40)(34+39)(35+38)(36+37)\\ &\times (42+51)(43+50)(44+49)(45+48)(46+47)\\ &\times (52+61)(53+60)(54+59)(55+58)(56+57)\\ &= 1 \times 13^5 \times 33^5 \times 53^5 \times 73^5 \times 93^5 \times 113^5.\end{aligned}$$

3. **Problem:** Find all real numbers x and y such that

$$\begin{aligned} x^2 + y^2 &= 2, \\ \frac{x^2}{2-y} + \frac{y^2}{2-x} &= 2. \end{aligned}$$

Solution: (Viet Hoang)
From the second equation, after a few steps of algebraic manipulation, one has

$$\begin{aligned} x^2(2-x) + y^2(2-y) &= 2(2-x)(2-y) \\ 2(x^2+y^2) - (x+y)(x^2+y^2-xy) &= 8 - 4(x+y) + 2xy \\ 4 - (x+y)(2-xy) &= 8 - 4(x+y) + 2xy \\ -2(x+y) + xy(x+y) &= 4 - 4(x+y) + 2xy \\ 2(x+y) + xy(x+y) &= 4 + 2xy \\ (x+y-2)(2+xy) &= 0 \end{aligned}$$

In the case $xy = -2$, substituting this into the first equation, we have $x^2 + xy + y^2 = 0$. However, this means

$$\left(x + \frac{y}{2}\right)^2 + \frac{3y^2}{4} = 0 \Longrightarrow x = y = 0$$

Substituting these values into the second equation, it yields a contradiction. For the second case, using the substitution $y = 2 - x$, we get

$$\begin{aligned} x^2 + y^2 &= 2, \\ x + y &= 2. \end{aligned}$$

From here we get: $(x-1)^2 + (y-1)^2 = (x^2+y^2) - 2(x+y) + 2 = 2 - 4 + 2 = 0$.
Therefore $x = y = 1$.

4. **Problem:** On a table, there is an empty bag and a chessboard containing exactly one token on each square. Next to the table is a large pile that contains an unlimited supply of tokens. Using only the following types of moves what is the maximum possible number of tokens that can be in the bag?

   - **Type 1:** Choose a non-empty square on the chessboard that is not in the rightmost column. Take a token from this square and place it, along with one token from the pile, on the square immediately to its right.
   - **Type 2:** Choose a non-empty square on the chessboard that is not in the bottom-most row. Take a token from this square and place it, along with one token from the pile, on the square immediately below it.
   - **Type 3:** Choose two adjacent non-empty squares. Remove a token from each and put them both into the bag.

   Solution A: (Ishan Nath)
   Let $a_{i,j}$ be the number of tokens in the square in the i^{th} row and j^{th} column, where the first row is the topmost row and the first column is the leftmost column. Furthermore let b denote the number of tokens in the bag. We define a monovariant as follows.

$$m = \frac{3b}{2} + \sum_{i=1}^{8}\sum_{j=1}^{8} 2^{16-i-j} a_{i,j}$$

   This either stays the same, or decreases after each move, because:

   - If Ross plays a type 1 move, then $a_{i,j}$ decreases by 1 and $a_{i,j+1}$ increases by 2 for some i, j. Thus m changes by

$$\begin{aligned} m' - m &= 2^{16-i-j}(-1) + 2^{16-i-(j+1)}(+2) \\ &= 0. \end{aligned}$$

     Hence m stays the same after type 1 moves.

   - If Ross plays a type 2 move, then $a_{i,j}$ decreases by 1 and $a_{i+1,j}$ increases by 2 for some i, j. Thus m changes by

$$\begin{aligned} m' - m &= 2^{16-i-j}(-1) + 2^{16-(i+1)-j}(+2) \\ &= 0. \end{aligned}$$

     Hence m stays the same after type 2 moves.

   - If Ross plays a type 3 move, then b increases by 2 while $a_{i,j}$ and $a_{i',j'}$ both decrease by 1 for some i, j, i', j'.

$$\begin{aligned} m' - m &= \frac{3}{2}(+2) + 2^{16-i-j}(-1) + 2^{16-i'-j'}(-1) \\ &\le \frac{3}{2}(+2) + 2^{0}(-1) + 2^{1}(-1) \\ &= 0. \end{aligned}$$

     Hence m stays the same or decreases after these moves.

At the beginning,

$$\begin{aligned}
m &= \frac{3b}{2} + \sum_{i=1}^{8}\sum_{j=1}^{8} 2^{16-i-j} a_{i,j} \\
&= 0 + \sum_{i=1}^{8}\sum_{j=1}^{8} 2^{16-i-j} \\
&= 0 + 2^{16} \sum_{i=1}^{8} 2^{-i} \sum_{j=1}^{8} 2^{-j} \\
&= 0 + 2^{16} \sum_{i=1}^{8} 2^{-i}(1-2^{-8}) \\
&= 0 + 2^{16}(1-2^{-8})(1-2^{-8}) \\
&= 65025.
\end{aligned}$$

Therefore at the end we must have $m \leq 65025$. Thus we obtain

$$\frac{3}{2}b \leq m \leq 65025 \quad \Longleftrightarrow \quad b \leq 43350.$$

To obtain equality we simply need to perform any strategy in which:

- Moves of type 3 are only ever performed on either $a_{7,8}$ and $a_{8,8}$, or $a_{8,7}$ and $a_{8,8}$.
- The chessboard is empty at the end of the game.

Now consider a strategy in which we start by applying all moves of type 1 and type 2, until $a_{i,j} = 0$ for all squares except for $a_{7,8}$, $a_{8,7}$ and $a_{8.8}$. At this point we will have $a_{8,8} = 1$ and since m is invariant,

$$65025 = 0 + a_{8,8} + 2a_{7,8} + 2a_{8,7} = 1 + 2(a_{7,8} + a_{8,7}).$$

Thus $a_{7,8} + a_{8,7} = 32512$. Now perform 10837 further moves (either type 1 on $a_{7,8}$ or type 2 on $a_{8,7}$) so that we end up with a state in which:

- $a_{7,8} + a_{8,7} = 32512 - 10837 = 21675$, and
- $a_{8,8} = 1 + 2 \times 10837 = 21675$.

From here we can finish by performing 21675 type 3 moves and obtain $b = 2 \times 21675 = 43350$ tokens in the bag.

Solution B: (Michael Albert)
First note that we may assume that all type 3 moves are performed after all the type 1 and type 2 moves have been made. WLOG We may also assume that all moves (type 1 and 2) into square $(8, 8)$ occur all other type 1 and type 2 moves.

Claim: In an optimal solution no type 3 move is made except from a pair including the lower right corner.

Proof: If we make one elsewhere, then we could first double each of those coins and move them to another adjacent pair of squares. Then we could perform two type 3 moves (in this new adjacent pair of squares) and get result in +2 more coins in the bag. □

From here it is clear that all the coins (except for the one that starts in $(8, 8)$) must eventually move to either $(7, 8)$ or $(8, 7)$. So there will be a point in which there is:

- one coin on $(8, 8)$,
- A coins on $(7, 8)$,
- B coins on $(8, 7)$, and
- all other squares are empty.

A coin starting in square (x, y) will need to double $(8 + 7 - x - y)$ times on it's way to either $(7, 8)$ or $(8, 7)$. Therefore

$$A + B = \sum_{(x,y)\neq(8,8)} 2^{8+7-x-y} = 2^{15} \sum_{x=1}^{8} 2^{-x} \sum_{y=1}^{8} 2^{-y} - 2^{-1} = 2^{-15} \left(\frac{2^{-9} - 2^{1}}{2^{-1} - 1} \right)^2 - \frac{1}{2} = 32512.$$

Now let m be the number of type 1 and type 2 moves we make before we finish the game by making as many type 3 moves as possible.

- After these m moves there will be $2m + 1$ coins on square $(8, 8)$.
- After these m moves there will be a total of $A + B - m = 32512 - m$ coins on squares $(7, 8)$ and $(8, 7)$.

So the maximum number of type 3 moves we can make is

$$M = \min (2m + 1, 32512 - m) .$$

If $m \leq 10837$ then $M = 2m + 1 \leq 21675$. If $m \geq 10837$ then $M = 32512 - m \leq 21675$. Hence the maximum occurs when $m = 10837$ and so $M = 21675$ type 3 moves are made.

5. **Problem:** A round-robin tournament is one where each team plays every other team exactly once. Five teams take part in such a tournament getting: 3 points for a win, 1 point for a draw and 0 points for a loss. At the end of the tournament the teams are ranked from first to last according to the number of points.

   (a) Is it possible that at the end of the tournament, each team has a different number of points, and each team except for the team ranked last has exactly two more points than the next-ranked team?

   (b) Is this possible if there are six teams in the tournament instead?

   Solution: (Ishan Nath)
   We show the answer is no for five teams, and yes for six.

   First, we show five teams cannot have this property. Suppose the teams had points $x, x+2, x+4, x+6, x+8$ in the final ranking. If there were d draws and $10-d$ decisive games, then the total number of points is

$$5x + 20 = x + (x+2) + \cdots + (x+8) = 2d + 3(10-d) = 30 - d.$$

   Since $0 \leq d \leq 10$, we have

$$20 \leq 5x + 20 \leq 30.$$

   Hence $x = 0, 1$ or 2.

   - If $x = 0$, then there are 20 points in total. Hence $d = 10$, so every match was a draw. However then every team would have 4 points, contradiction.
   - If $x = 2$, then there are 30 points in total. Hence $d = 0$, so every match was decisive. But this means every team should have a point total which is a multiple of 3, since they can earn either 0 or 3 points per game. However the lowest team has 2 points, contradiction.
   - If $x = 1$, then there are 25 points in total. Hence $d = 5$, meaning there are 5 draws and 5 decisive games. In this case, the teams had points $1, 3, 5, 7, 9$.

     Consider the total number of draws. For each team, the number of points satisfies $p = 3w + d$, where p is the number of points, w is the number of wins, and d is the number of draws. Hence the number of draws is equal to the number of points, modulo 3.

     – The team with 1 points has drawn at most once

     – The team with 3 points has drawn at most 3 times.

     – The team with 5 points has drawn at most 2 times, since they play 4 times.

     – The team with 7 points has drawn at most once. They cannot draw 4 times, otherwise they would have drawn every game, ending up with 4 points.

     – The team with 9 points cannot have drawn. Otherwise, they would have drawn at least 3 times, requiring 6 points from the remaining game, which is impossible.

     Hence there are at most 7 draws. However, if 5 matches ended in draws, then there should be at least 10 draws, contradiction.

   All of these options lead to contradictions, so the rankings cannot have each team having exactly two more points than the least.

With six teams it is possible: Take 6 teams, labeled 1 to 6. We assign the following matches:

- Teams 1 and 2 draw, teams 1 and 4 draw, and teams 4 and 5 draw.
- Team 2 beats team 6.
- Otherwise, if $1 \leq x < y \leq 6$, then team y beats team x.

Then team i has $2i$ points, satisfying the condition.

6. **Problem:** Let a positive integer n be given. Determine, in terms of n, the least positive integer k such that among any k positive integers, it is always possible to select a positive even number of them having sum divisible by n.

Solution: (Ethan Ng)
First we consider the n odd case. If n is odd we claim the answer is $k = 2n$.
Let the numbers be $x_1, x_2, x_3, \ldots, x_{2n}$. Consider the following partial sums:

$$\begin{aligned} s_1 &= x_1 + x_2 \\ s_2 &= x_1 + x_2 + x_3 + x_4 \\ &\vdots \\ s_i &= x_1 + x_2 + x_3 + \cdots + x_{2i} \\ &\vdots \\ s_n &= x_1 + x_2 + x_3 + x_4 + \cdots + x_{2n}. \end{aligned}$$

If we have $s_i \equiv 0 \pmod{n}$ for any i, then we are done. Otherwise, by the Pigeonhole Principle we must have a pair of distinct indices $1 \le i < j \le n$ such that $s_i \equiv s_j \pmod{n}$. Thus we can construct:

$$x_{2i+1} + x_{2i+2} + \cdots + x_{2j} = s_j - s_i \equiv 0 \pmod{n}$$

as required.

However if $k < 2n$ then we could choose: $x_1 = x_2 = \cdots = x_k = 1$ and so the sum of any non-empty subset would be a positive integer less than $2n$. So if this sum is a multiple of n then it must be equal to n, and so we must have selected n of them. But n is odd, so it is impossible to select an even number of them having sum divisible by n.

Now we consider the n even case. If n is even we claim the answer is $k = n + 1$.
Let $n = 2m$. By the Pigeonhole Principle, amongst any three positive integers we can choose two of them with the same parity. Therefore we can order the $k = 2m+1$ integers: $x_1, x_2, x_3, \ldots, x_{2m+1}$ such that

$$a_1 = \frac{x_1 + x_2}{2}, \; a_2 = \frac{x_3 + x_4}{2}, \; \ldots \; a_m = \frac{x_{2m-1} + x_{2m}}{2}$$

are all integers. This can be done by iteratively choosing x_{2i-1} and x_{2i} so that they have the same parity. Now consider the partial sums:

$$\begin{aligned} t_1 &= a_1 \\ t_2 &= a_1 + a_2 \\ t_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ t_m &= a_1 + a_2 + a_3 + \cdots + a_m. \end{aligned}$$

If any t_i were a multiple of m then we would have $2t_i = x_1 + x_2 + x_3 + \cdots + x_{2i}$ being a multiple of $2m$. Otherwise consider the m pigeons $(t_1, \ldots t_m)$ and the $m-1$ pigeonholes (the non-zero residues mod m). By the Pigeonhole Principle we can find indices $1 \le i < j \le m$ such that $t_i \equiv t_j \pmod{m}$. Therefore

$$a_{i+1} + a_{i+2} + \cdots + a_j = t_j - t_i \equiv 0 \pmod{m}.$$

Thus $(x_{2i+1} + \cdots + x_{2j})/2$ is a multiple of m and hence $x_{2i+1} + \cdots + x_{2j}$ is a multiple of $2m = n$.
However if $k \le n$ then we could choose: $x_1 = n$ and $x_2 = \cdots = x_k = 1$ and so the sum of any non-empty subset would be a positive integer at most $2n - 1$. So if this sum is a multiple of n then it must be equal to n. However the only way to achieve a sum of n is to only select x_1 which is not selecting an even number of numbers.

Comment It is possible to come up with optimal constructions in which the (x_i) are all different. For example when n is odd choose:

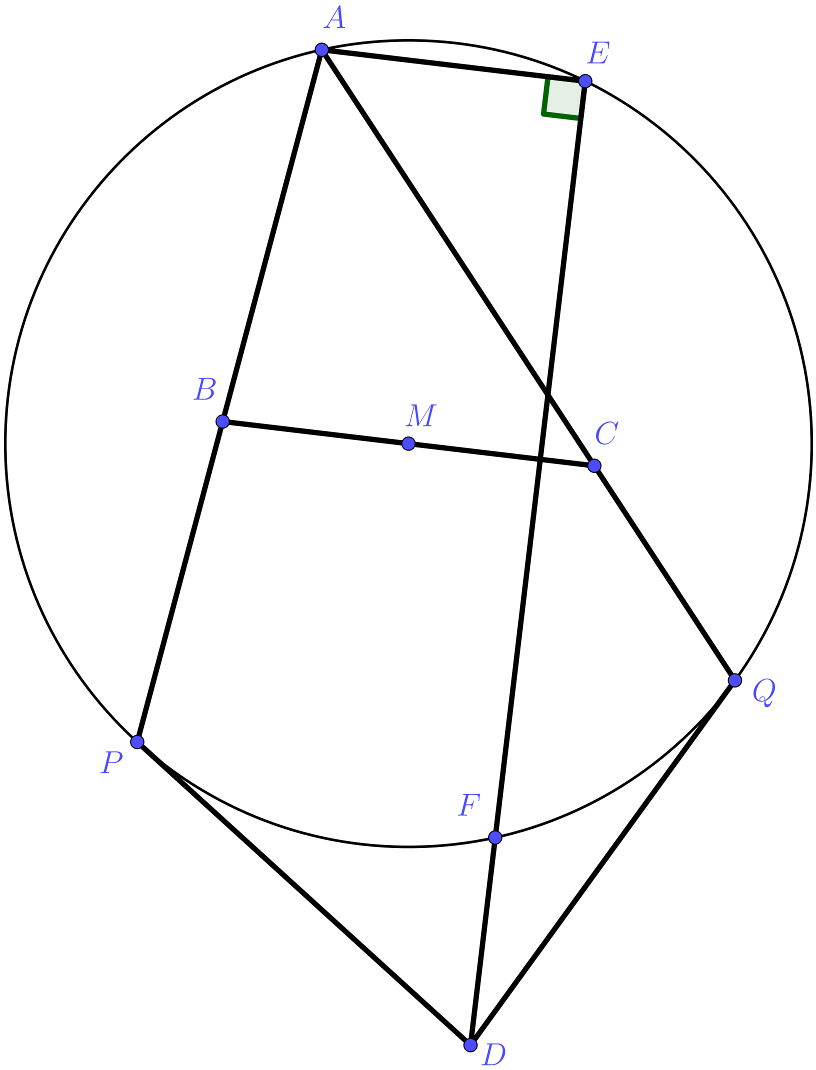
$$(x_1, x_2, x_3, \ldots, x_k) = (n+1, 2n+1, 3n+1, \ldots, kn+1).$$

And when n is even we can choose:

$$(x_1, x_2, x_3, \ldots, x_k) = (n, 2n+1, 3n+1, \ldots, kn+1).$$

7. **Problem:** Let M be the midpoint of side BC of acute triangle ABC. The circle centered at M passing through A intersects the lines AB and AC again at P and Q, respectively. The tangents to this circle at P and Q meet at D. Prove that the perpendicular bisector of BC bisects segment AD.

Solution A: (Ross Atkins)
Let F be the point on the circle such that AF is a diameter. So M is the midpoint of AF. Now let $E \neq F$ be the point on the circle such that D, E, F are colinear. This means $\angle AEF = 90°$ by Thales' theorem.



By the Power of a Point Theorem we get $DP^2 = DE \times DF = DQ^2$. Therefore we have two pairs of similar triangles: $EPD \sim PFD$ and $EQD \sim QFD$. Hence

$$\frac{EP}{PF} = \frac{PD}{FD} = \frac{QD}{FD} = \frac{EQ}{QF}.$$

i.e. $EPFQ$ is a harmonic quadrilateral. Now apply the extended sine rule to get:

$$\frac{\sin \angle BAM}{\sin \angle MAC} = \frac{\sin \angle PAF}{\sin \angle FAQ} = \frac{PF}{FQ} = \frac{PE}{EQ} = \frac{\sin \angle PAE}{\sin \angle QAE} = \frac{\sin \angle BAE}{\sin \angle CAE}.$$

Since $\frac{\sin \angle BAM}{\sin \angle MAC} = \frac{\sin \angle BAE}{\sin \angle CAE}$, and M is the midpoint of BC, this means AE is parallel to BC. Hence the perpendicular bisector of BC is parallel to line DEF.
Now let N be the midpoint of AD. By midpoint theorem in triangle AFD we get $NM \parallel DF$ and therefore N lies on the perpendicular bisector of BC.

Solution B: (Ishan Nath)
As in the previous solution, let F be the point on the circle such that AF is a diameter. Notice that

$$\angle BPF = 90^\circ = \angle MPD,$$

where the first equality follows from Thales' theorem, and the second follows as PD is tangent to a circle with center M. We also have

$$\angle PBF = \angle BAC = \angle PAQ = \frac{\angle PMQ}{2} = \angle PMD,$$

since $CABF$ is a parallelogram (because the diagonals bisect each other at M) and $\angle PMQ = 2\angle PMD$ (because $PMQD$ is a kite). This implies that the triangles PBF and PMD are similar. Since they are also similarly oriented, we can use similar switch to deduce that triangles PBM and PFD are similar.

Consider rotation at P, followed by scaling, that takes triangle PBM to PFD. Since $\angle BPF = 90^\circ$, the rotation must be 90°. However this means the transformation taking BM to FD must rotate BM by 90°, implying BM and FD are perpendicular.

Finally, letting N be the midpoint of AD, we get $NM \parallel DF \perp BC$ by the midpoint theorem. So N lies on the perpendicular bisector of BC.

8. **Problem:** Find all real numbers x such that $-1 < x \leq 2$ and

$$\sqrt{2-x} + \sqrt{2+2x} = \sqrt{\frac{x^4+1}{x^2+1}} + \frac{x+3}{x+1}.$$

Solution: (Viet Hoang)
Notice that we are solving for x in the domain $-1 < x \leq 2$. Using Cauchy-Schwarz Inequality on the left hand side, one has

$$\sqrt{2-x} + \sqrt{2+2x} = \sqrt{\frac{1}{2} \cdot (4-2x)} + \sqrt{2+2x} \leq \sqrt{\left(\frac{1}{2}+1\right)(2+2x+4-2x)} = 3.$$

Using the fact that $x^4 + 1 \geq 2x^2$ (because $(x^2-1)^2 \geq 0$) and $x^2 + 1 \geq 2x$ (because $(x-1)^2 \geq 0$ we can get:

$$2(x^4+1) \geq (x^2+1)^2 \geq (x^2+1)\frac{(x+1)^2}{2}.$$

If we substitute this into the right-hand side we get

$$\sqrt{\frac{x^4+1}{x^2+1}} + \frac{x+3}{x+1} \geq \frac{x+1}{2} + \frac{x+3}{x+1} = \frac{x+1}{2} + \frac{2}{x+1} + 1.$$

Finally applying the AM-GM again for 2 positive numbers $\frac{x+1}{2}$ and $\frac{2}{x+1}$ we obtain $\frac{x+1}{2} + \frac{2}{x+1} \geq 2$. Thus the right-hand side is ≥ 3.

Therefore, we can see that equality must occur in our AM-GM inequality and hence, $\dfrac{x+1}{2} = \dfrac{2}{x+1} \Longrightarrow x = 1$.