# New Zealand Mathematical Olympiad Committee

## NZMO Round One 2021 — Solutions

1. **Problem:** A school offers three subjects: Mathematics, Art and Science. At least 80% of students study both Mathematics and Art. At least 80% of students study both Mathematics and Science. Prove that at least 80% of students who study both Art and Science, also study Mathematics.

   **Solution:** Let $n$ be the total number of students. Let $x$ be the number of students that study all three subjects. Let $a$ be the number of students that study Maths and Art but not Science. Let $b$ be the number of students that study Maths and Science but not Art. Let $c$ be the number of students that study Art and Science but not Maths.
   So $a, b, c, x, n$ are non-negative real numbers with

   - $n \geq a + b + c + x$ (total number of students).
   - $x + a \geq 0.8 \times n$ (studying both Mathematics and Art),
   - $x + b \geq 0.8 \times n$ (studying both Mathematics and Science).

   Adding all these together gives us $2x + a + b \geq 1.6 \times n$ and since $n \geq a + b + c + x$ we get:

   $$2x + a + b \geq 1.6 \times (a + b + c + x).$$

   Finally multiply both sides by 0.5 and rearrange to get:

   $$x \geq 0.8 \times (x + c) + 0.3 \times (a + b) \geq 0.8 \times (x + c).$$

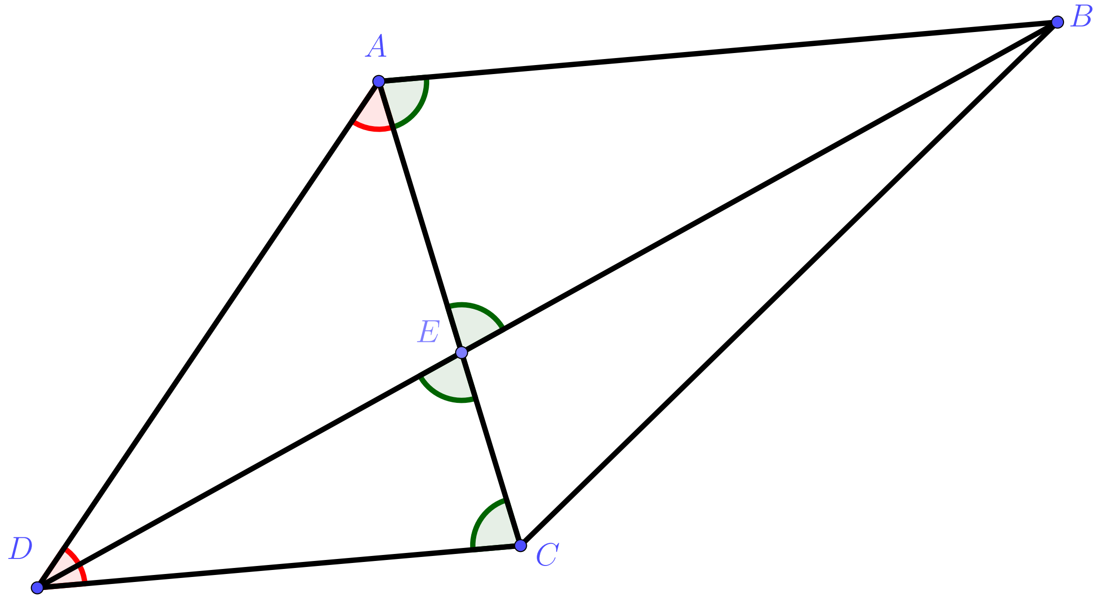
   Therefore $\frac{x}{x+c} \geq 0.8$ as required.

   **Alternative Solution:**
   First note that if $x \geq 0.8n$ then $\frac{x}{x+c} \geq \frac{x}{n} \geq 0.8$. Otherwise we have $x < 0.8n$ and so $n > \frac{5x}{4}$. Hence

   $$\begin{aligned} \frac{x}{x+c} &= \frac{x}{(a+b+c+x) - (a+x) - (b+x) + 2x} \\ &\geq \frac{x}{n - 0.8n - 0.8n + 2x} \\ &= \frac{x}{2x - 0.6n} \\ &> \frac{x}{2x - 0.6 \times \frac{5x}{4}} \\ &= 0.8. \end{aligned}$$

2. **Problem:** Let $ABCD$ be a trapezium such that $AB \parallel CD$. Let $E$ be the intersection of diagonals $AC$ and $BD$. Suppose that $AB = BE$ and $AC = DE$. Prove that the internal angle bisector of $\angle BAC$ is perpendicular to $AD$.

**Solution:** First note that triangle $ABE$ is isosceles because $AB = BE$.




Let $x = \angle DEC$. Angle chasing gives:

$$\begin{aligned} x &= \angle DEC \\ &= \angle BEA && \text{(vertically opposite)} \\ &= \angle EAB && (\triangle ABE \text{ isosceles}) \\ &= \angle ACD. && (AB \text{ parallel } CD) \end{aligned}$$

Therefore triangle $CDE$ is isosceles. Hence $DE = DC$. Since we are also given $AC = DE$ this implies $AC = DC$. Therefore $\triangle ACD$ is isosceles. Since $x = \angle ACD$ this gives us

$$\angle CDA = \angle DAC = 90^\circ - \frac{x}{2}$$

Now let $\lambda$ be the angle bisector of $\angle BAC$. Since $x = \angle BAC$ we know that $\lambda$ makes an angle of $\frac{1}{2}\angle BAC = \frac{x}{2}$ with line $AC$. Therefore the angle between $\lambda$ and $AD$ is

$$\frac{x}{2} + \angle DAC = \frac{x}{2} + \left(90^\circ - \frac{x}{2}\right) = 90^\circ$$

as required.

3. **Problem:** In a sequence of numbers, a term is called *golden* if it is divisible by the term immediately before it. What is the maximum possible number of golden terms in a permutation of $1, 2, 3, \ldots, 2021$?

**Solution:** Let $k$ be the number of golden terms. We claim that $k \leq 1010$.
Proof: Define the term immediately before a golden term to be a *silver* term. The number of silver terms is also $k$. If $a$ is any silver term and $b$ is the corresponding golden term then we must have

$$a \leq \frac{b}{2} \leq \frac{2021}{2} < 1011.$$

Therefore $a \leq 1010$. Hence $a \in \{1, 2, 3, \ldots, 1010\}$ and therefore there can be at most 1010 different silver terms. Therefore $k \leq 1010$.

Now we need to show that $k = 1010$ is indeed possible. To do this we first partition the set $\{1, 2, 3, \ldots, 2021\}$ into the following distinct parts:

$$\begin{aligned}
&\{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\},\\
&\{3, 6, 12, 24, 48, 96, 192, 384, 768, 1536\},\\
&\{5, 10, 20, 40, 80, 160, 320, 640, 1280\},\\
&\{7, 14, 28, 56, 112, 224, 448, 896, 1792\},\\
&\ldots\\
&\{2019\},\\
&\{2021\}.
\end{aligned}$$

Each part starts with an odd number, which is then doubled until the result is larger than 2021.
Putting each part in the above sequence in order is our example. In this example every even term, $2n$, is golden because it occurs immediately after the term $n$. There are 1010 even numbers, so $k = 1010$ is possible.

**Comment:**
A full solution needs to show that *any* permutation cannot have 1011 (or more) golden terms. Simply claiming that a particular permutation is "optimal" or "best" would not receive full marks for this problem. There are many permutations with 1010 golden terms such that not every golden term is double it's predecessor. For example:

$$\begin{aligned}
&4, 8, 16, 32, 64, 128, 256, 512, 1024,\\
&1, 2, 6, 12, 24, 48, 96, 192, 384, 768, 1536,\\
&5, 10, 20, 40, 80, 160, 320, 640, 1280,\\
&7, 14, 28, 56, 112, 224, 448, 896, 1792,\\
&3, 9, 18, 36, 72, 144, 288, 576, 1152,\\
&11, \ldots\\
&2021.
\end{aligned}$$

4. **Problem:** Find all triples $(x, p, n)$ of non-negative integers such that $p$ is prime and

$$2x(x+5) = p^n + 3(x-1).$$

**Solution:** The equation rearranges to be

$$p^n = 2x(x+5) - 3(x-1) = 2x^2 + 7x + 3 = (2x+1)(x+3).$$

Since $x$ is a non-negative integer, both factors $(2x+1)$ and $(x+3)$ must be positive integers. Therefore both $(2x+1)$ and $(x+3)$ are both powers of $p$. Let

$$\begin{aligned} 2x+1 &= p^a \\ x+3 &= p^b. \end{aligned}$$

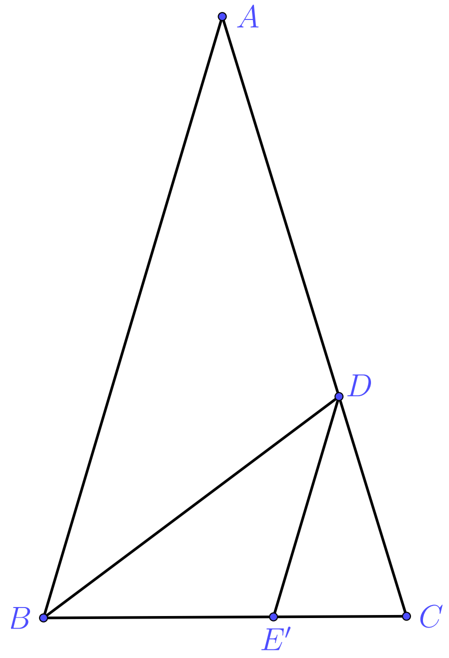
Now note that $\gcd(2x+1, x+3) = \gcd(x-2, x+3) = \gcd(5, x+3)$ which must equal either 1 or 5 (because 5 is prime). We consider each case individually:

- If $\gcd(2x+1, x+3) = 5$ then $p = 5$ and $\min(a, b) = 1$. Thus either $2x+1 = 5$ or $x+3 = 5$. Either way we get $x = 2$. This leads to $p^n = 25$ and so $(x, p, n) = (2, 5, 2)$.
- If $\gcd(2x+1, x+3) = 1$ then $\min(a, b) = 0$. Thus $2x+1 = 1$ or $x+3 = 1$. So either $x = 0$ or $x = -2$. We can't have $x < 0$ so we must have $x = 0$. This leads to $p^n = 3$ and so $(x, p, n) = (0, 3, 1)$.

Therefore the only solutions for $(x, p, n)$ are $(2, 5, 2)$ and $(0, 3, 1)$.

5. **Problem:** Let $ABC$ be an isosceles triangle with $AB = AC$. Point $D$ lies on side $AC$ such that $BD$ is the angle bisector of $\angle ABC$. Point $E$ lies on side $BC$ between $B$ and $C$ such that $BE = CD$. Prove that $DE$ is parallel to $AB$.

   **Solution:** Let $E'$ be the point on line $BC$ such that $DE'$ is parallel to $AB$. We know that $E'$ lies between $B$ and $C$ because $D$ lies between $A$ and $C$. So it suffices for us to prove that $BE' = CD$.




$$\begin{aligned} \angle ACB &= \angle CBA && (\triangle ABC \text{ isosceles}) \\ &= \angle CE'D. && (AB \parallel DE') \end{aligned}$$

Therefore triangle $CDE'$ is isosceles with $CD = DE'$.

$$\begin{aligned} \angle E'BD &= \angle DBA && (BD \text{ bisects } \angle CBA) \\ &= \angle ADE'. && (AB \parallel DE') \end{aligned}$$

Therefore triangle $BE'D$ is isosceles with $BE' = DE'$.
Hence $CD = DE' = BE'$ as required.

6. **Problem:** Is it possible to place a positive integer in every cell of a $10 \times 10$ array in such a way that both the following conditions are satisfied?

- Each number (not in the top row) is a proper divisor of the number immediately below.
- Each row consists of 10 consecutive positive integers (but not necessarily in order).

**Solution:** Answer: Yes. In fact it is even possible to achieve such an array where each row consists of ten consecutive positive integers in increasing order. We shall construct an example explicitly.
Initially let the top row be $(1, 2, 3, \ldots, 10)$ in this order. Then iteratively if the contents of a particular row are

$$\big(n, n+1, n+2, \ldots, n+9\big)$$

then construct the next row to be

$$\big((n+9)! + n, (n+9)! + n + 1, (n+9)! + n + 2, \ldots, (n+9)! + n + 9\big).$$

So the completed array will look like this:

| | | | | |
|---|---|---|---|---|
| $1$ | $2$ | $3$ | $\cdots$ | $10$ |
| $10! + 1$ | $10! + 2$ | $10! + 3$ | $\cdots$ | $10! + 10$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| $n$ | $n+1$ | $n+2$ | $\cdots$ | $n+9$ |
| $(n+9)! + n$ | $(n+9)! + n + 1$ | $(n+9)! + n + 2$ | $\cdots$ | $(n+9)! + n + 9$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

To prove that this construction works, we simply need to verify that $((n+9)! + n + i)$ is always a multiple of $(n+i)$ for each $i = 0, 1, 2, 3, \ldots, 9$. We prove this by combining these two trivial divisibilities:

$$(n+i) \mid (n+9)! \qquad \text{and} \qquad (n+i) \mid (n+i).$$

Thus this construction satisfies the required conditions.

**Comment:**
The construction given here can be generalised in the following way. Let the first row be any arrangement of 10 consecutive positive integers. Then iteratively if the contents of a particular row is $(a_1, a_2, \ldots, a_{10})$, then choose some positive integer $x$ which is a common multiple of each of $a_1, a_2, \ldots, a_{10}$ and construct the next row to be $(b_1, b_2, \ldots, b_{10})$, where $b_i = x + a_i$ for all $i$. Note that we cannot simply let $x = 0$ because a number is not a *proper divisor* of itself.

7. **Problem:** Let $a, b, c, d$ be integers such that $a > b > c > d \geq -2021$ and

$$\frac{a+b}{b+c} = \frac{c+d}{d+a}$$

(and $b + c \neq 0 \neq d + a$). What is the maximum possible value of $ac$?

**Solution:** We claim that the maximum value of $ac$ is $2 \times 505^2 = 510050$, and this is uniquely achieved by $(a, b, c, d) = (1010, 506, 505, -2021)$.
To prove this we start by rearranging the expression $\frac{a+b}{b+c} = \frac{c+d}{d+a}$ to get $(c + b)(c + d) = (a + b)(a + d)$. Now expand the brackets and move all the terms to one side to get $(c - a)(a + b + c + d) = 0$. Since $c > a$ we can divide by $c - a$ to get

$$a + b + c + d = 0.$$

Note that this is equivalent to the $\frac{a+b}{b+c} = \frac{c+d}{d+a}$ condition. Now assume that $(a, b, c, d)$ is the tuple such that $ac$ is maximised. Since $4a > a + b + c + d = 0$ we trivially have $a > 0$. Furthermore if $c \leq 0$ then we would have $ac \leq 0$ which would contradict the definition that $ac$ is maximal. Therefore

$$a > b > c > 0 \geq d \geq -2021.$$

Now we perform two little proofs by contradiction:

- If $d > -2021$ then we could consider $(a', b', c', d') = (a + 1, b, c, d - 1)$ so that

  $$a'c' = (a + 1)c > ac.$$

  However this would contradict the definition that $ac$ was maximal.
  Therefore $d = -2021$.

- If $b > c + 1$ then we could consider $(a', b', c', d') = (a + 1, b - 1, c, d)$ so that

  $$a'c' = (a + 1)c > ac.$$

  However this would contradict the definition that $ac$ was maximal.
  Therefore $b = c + 1$.

Therefore $(a, b, c, d) = (a, c + 1, c, -2021)$, and since $a + b + c + d = 0$, this means $a = 2020 - 2c$. So the final expression we are trying to maximise is

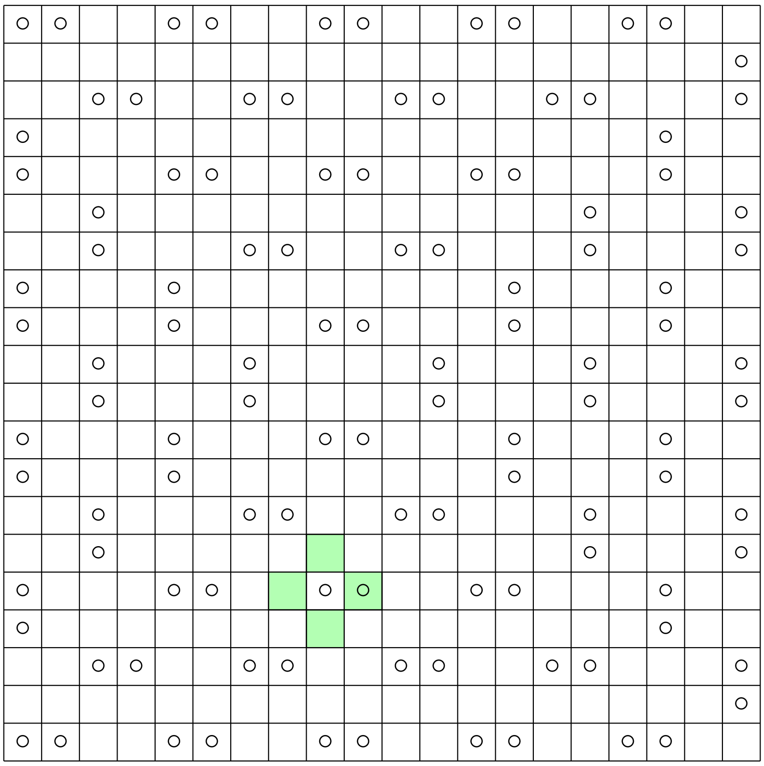
$$\begin{aligned} ac &= (2020 - 2c)c \\ &= 2 \times 505^2 - 2(c - 505)^2 \\ &\geq 2 \times 505^2. \end{aligned}$$

with equality iff $c = 505$. Therefore the maximum value of $ac$ is $2 \times 505^2 = 510050$ which is achieved by

$$(a, b, c, d) = (1010, 506, 505, -2021).$$

8. **Problem:** Two cells in a $20 \times 20$ board are *adjacent* if they have a common edge (a cell is not considered adjacent to itself). What is the maximum number of cells that can be marked in a $20 \times 20$ board such that every cell is adjacent to at most one marked cell?

**Solution:** In the following diagram, there are 110 cells containing an $\circ$ symbol.



Notice that each cell is adjacent to exactly one $\circ$ (including the cells containing a $\circ$). Now partition all 400 cells into 110 pigeon-holes based on which $\circ$ the cell is adjacent to.

Two cells are in the same pigeon-hole if and only iff they are adjacent to the same $\circ$.

For example the four coloured cells are in the same pigeon-hole. By the constraint given in the problem, we cannot mark two cells (or more) in the same pigeon-hole. Since there are 110 pigeon-holes, this means we can mark at most 110 cells. Therefore it is impossible to mark more than 110 cells.

In order to demonstrate that it is possible to mark 110 cells, we provide a construction. One suitable construction is to simply mark each of the cells in the previous diagram containing a $\circ$.