New Zealand Mathematical Olympiad Committee

NZMO Round One 2020 — Solutions

1. **Problem:** What is the maximum integer n such that $\frac{50!}{2^n}$ is an integer?

   Solution: $50! = 1 \times 2 \times 3 \times \cdots \times 50$. Of the numbers up to 50, we need to find how many of them are divisible by 2^k for each $k = 1, 2, 3, 4, 5, 6$. There are $\lfloor \frac{50}{2^k} \rfloor$ numbers which are divisible by 2^k and there are $\lfloor \frac{50}{2^{k+1}} \rfloor$ which are divisible by 2^{k+1}. Therefore there are

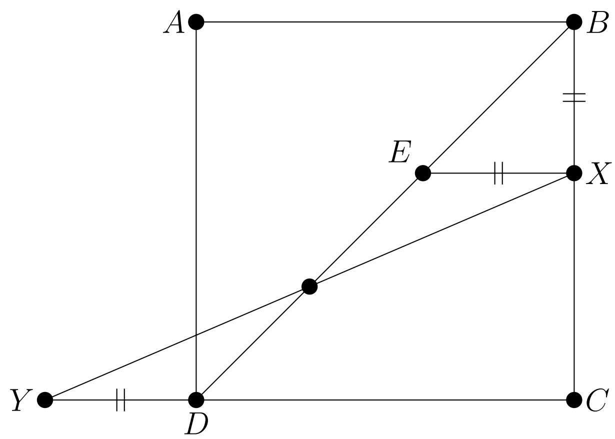
   $$\left\lfloor \frac{50}{2^k} \right\rfloor - \left\lfloor \frac{50}{2^{k+1}} \right\rfloor$$

   numbers between 1 and 50 which are divisible by 2 exactly k times. Hence the answer we are looking for is:

   $$\left(\lfloor \tfrac{50}{2} \rfloor - \lfloor \tfrac{50}{4} \rfloor\right) + 2\left(\lfloor \tfrac{50}{4} \rfloor - \lfloor \tfrac{50}{8} \rfloor\right) + 3\left(\lfloor \tfrac{50}{8} \rfloor - \lfloor \tfrac{50}{16} \rfloor\right) + 4\left(\lfloor \tfrac{50}{16} \rfloor - \lfloor \tfrac{50}{32} \rfloor\right) + 5\left(\lfloor \tfrac{50}{32} \rfloor - \lfloor \tfrac{50}{64} \rfloor\right)$$
   $$= 13 + 2 \times 6 + 3 \times 3 + 4 \times 2 + 5 \times 1$$
   $$= 47.$$

2. **Problem:** Let $ABCD$ be a square and let X be any point on side BC between B and C. Let Y be the point on line CD such that $BX = YD$ and D is between C and Y. Prove that the midpoint of XY lies on diagonal BD.

   Solution:



   Construct point E on diagonal BD so that EX is parallel to CD. So $\angle BXE = \angle BCD = 90°$. Also $\angle XBE = 45°$ because E is on diagonal BD. Therefore triangle $\triangle BEX$ is an isosceles right-angled triangle. Hence

   $$EX = BX = YD.$$

   Since segments EX and YD are parallel and equal in length, this implies that $EXDY$ is a parallelogram. Since the diagonals of a parallelogram bisect each other, we deduce that the intersection of XY and DE is the midpoint of XY. Therefore the midpoint of XY lies on line DE (which is a segment of diagonal BD).

Alternative Solution A: Let $ABCD$ be the unit square, with $A = (0,1)$, $B = (1,1)$, $C = (1,0)$ and $D = (0,0)$. Now let $a = BX = DY$. This means that $X = (1, 1-a)$ and $Y = (-a, 0)$. Now let Z be the midpoint of XY. We compute the co-ordinates of Z to be

$$Z = \left(\frac{1+(-a)}{2}, \frac{(1-a)+0}{2}\right).$$

The x and y coordinates of Z are equal, therefore Z lies on diagonal BD.

Alternative Solution B: Let s be the side-length of the square and let $a = BX = DY$. Let Z be the midpoint of XY. Now we apply the converse of Menelaus' Theorem to traversal BZD of $\triangle XYC$.

$$\frac{XZ}{ZY} \times \frac{YD}{DC} \times \frac{CB}{BX} = 1 \times \frac{a}{s} \times \frac{s}{a} = 1.$$

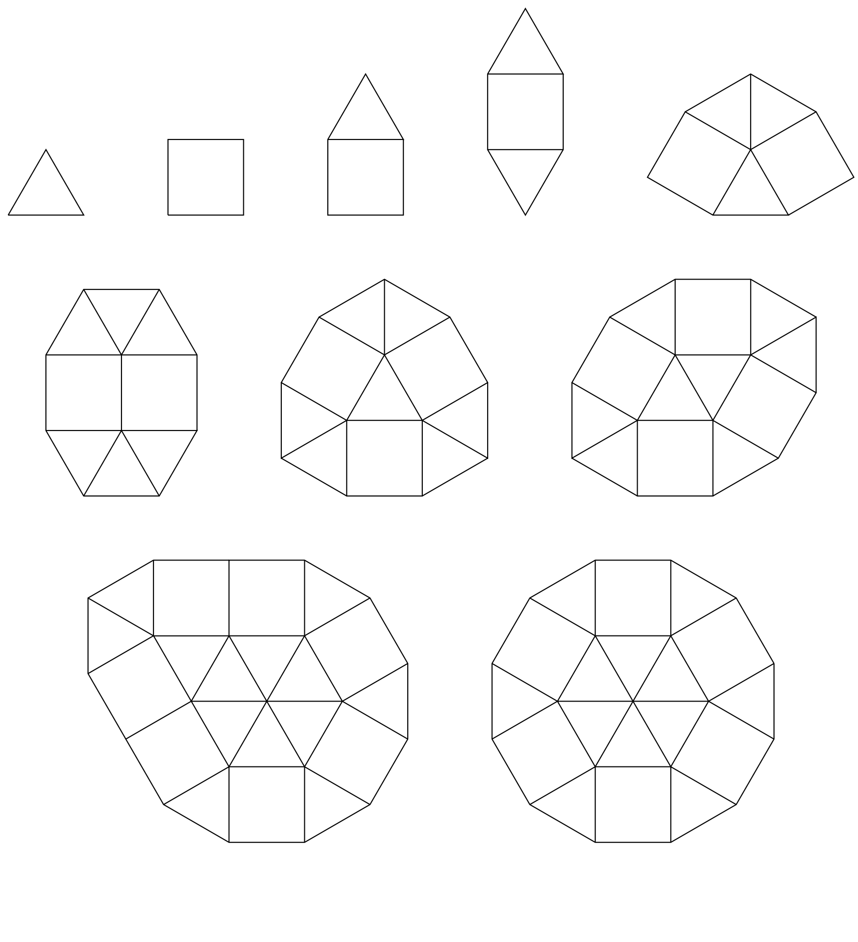
Therefore Z, D and B are colinear.

3. **Problem:** You have an unlimited supply of square tiles with side length 1 and equilateral triangle tiles with side length 1. For which n can you use these tiles to create a convex n-sided polygon? The tiles must fit together without gaps and may not overlap.

Solution: All the angles in squares and equilateral triangles are multiples of 30°. So all the external angles of the n-sided polygon are multiples of 30°. Since the polygon is convex, this implies that all external angles are greater than or equal to 30°. However, the sum of the external angles is 360°, therefore

$$n \times 30° \leq 360°.$$

Hence $n \leq 12$. Also all polygons have at least 3 sides so $3 \leq n \leq 12$. Finally we demonstrate that it is possible for any $3 \leq n \leq 12$ using the following illustrations.



4. **Problem:** Determine all prime numbers p such that $p^2 - 6$ and $p^2 + 6$ are both prime numbers.

   Solution: If $p > 5$ then the units digit of p must be 1, 3, 7 or 9.

   - If the units digit of p is 1 or 9 then the units digit of p^2 is 1. Therefore the units digit of $p^2 - 6$ is 5. Since $p^2 - 6 > 5$ this means that $p^2 - 6$ is not prime.
   - If the units digit of p is 3 or 7 then the units digit of p^2 is 9. Therefore the units digit of $p^2 + 6$ is 5. Since $p^2 + 6 > 5$ this means that $p^2 + 6$ is not prime.

   Therefore we must have $p \leq 5$. Hence p must be 2, 3 or 5.

   - If $p = 2$ then $p^2 + 6 = 10$ is not prime.
   - If $p = 3$ then $p^2 + 6 = 15$ is not prime.
   - If $p = 5$ then $p^2 \pm 6$ are 19 and 31 which are both prime.

   Therefore the only answer is $p = 5$.

   Alternative Solution: Consider the following product modulo 5.

   $$p(p^2 - 6)(p^2 + 6) = p^5 - 36p \equiv p^5 - p \pmod 5$$

   By Fermat's Little Theorem, this product is 0 (mod 5). So if p, $p^2 - 6$ and $p^2 + 6$ are all prime numbers then at least one of them must be equal to 5.

   - If $p = 5$ then $6p^2 - 1 = 29$ and $6p^2 + 1 = 31$. This is one solution.
   - If $6p^2 - 1 = 5$ then $p = \pm 1$. Neither 1 nor -1 is prime, so this case leads to no solutions.
   - If $6p^2 + 1 = 5$ then p is not an integer. No solutions in this case.

   Therefore the only solution is $p = 5$.

5. **Problem:** Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

   $$f(x + f(y)) = 2x + 2f(y + 1)$$

   for all real numbers x and y.

   Solution: Substitute $x = -f(y)$ into the given equation to get

   $$f(0) = -2f(y) + 2f(y + 1) \tag{1}$$

   We can then substitute $y = -1$ into (1) to get $f(0) = -2f(-1) + 2f(0)$. Hence

   $$f(-1) = \frac{f(0)}{2}. \tag{2}$$

   Also we can substitute $y = -2$ into (1) to get $f(0) = -2f(-2) + 2f(-1)$. Hence

   $$f(-1) = f(-2) + \frac{f(0)}{2}. \tag{3}$$

   Equations (2) and (3) tell us that $f(-2) = 0$. Now substitute $y = -2$ into the original equation to get

   $$f(x + f(-2)) = 2x + 2f(-1) \quad \Longrightarrow \quad f(x) = 2x + 2f(-1).$$

Therefore our function can be written in the form $f(x) = 2x + c$, where $c = 2f(-1)$ is a constant. We can substitute this into the original equation to get

$$\begin{aligned} & f(x + f(y)) = 2x + 2f(y + 1) \\ \iff \quad & f(x + (2y + c)) = 2x + 2(2(y + 1) + c) \\ \iff \quad & 2(x + (2y + c)) + c = 2x + 4y + 4 + 2c \\ \iff \quad & 2x + 4y + 3c = 2x + 4y + 2c + 4 \\ \iff \quad & c = 4. \end{aligned}$$

Therefore the function f works if and only if $c = 4$. Hence $f(x) = 2x + 4$ (for all x) is the only solution.

Alternative Solution:
Substitute $x = z - f(0)$ and $y = 0$ into the given equation to get

$$\begin{aligned} f((z - f(0)) + f(0)) &= 2(z - f(0)) + 2f(1) \\ f(z) &= 2z + (2f(1) - 2f(0)). \end{aligned}$$

Hence $f(z) = 2z + c$ for any real number z, where $c = 2f(1) - 2f(0)$ is constant. Now

$$f(1) - f(0) = (2 \cdot 1 + c) - (2 \cdot 0 + c) = 2.$$

Hence $c = 2f(1) - 2f(0) = 4$. Therefore $f(z) = 2z + 4$ is the only possible function satisfying the given equation.

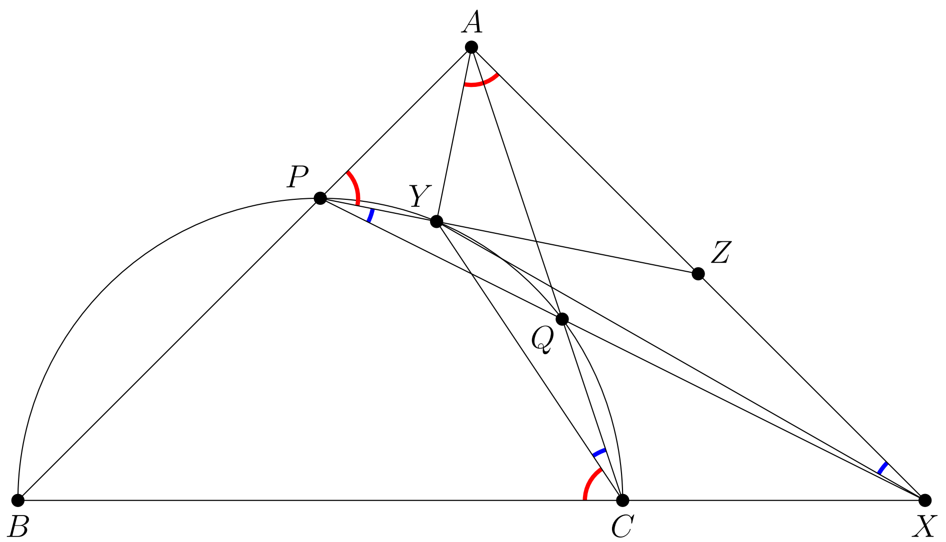
Now need to check that $f(z) = 2z + 4$ does indeed satisfy the given equation

$$\begin{aligned} f(x + f(y)) = f(x + 2y + 4) = 2(x + 2y + 4) + 4 &= 2x + 4y + 12 \\ &= 2x + 2(2(y + 1) + 4) = 2x + 2f(y + 1). \end{aligned}$$

Hence $f(z) = 2z + 4$ is the unique function f satisfying the given equation.

6. **Problem:** Let $\triangle ABC$ be an acute triangle with $AB > AC$. Let P be the foot of the altitude from C to AB and let Q be the foot of the altitude from B to AC. Let X be the intersection of PQ and BC. Let the intersection of the circumcircles of triangle $\triangle AXC$ and triangle $\triangle PQC$ be distinct points: C and Y. Prove that PY bisects AX.

Solution: Let Z be the point where PY intersects AX. The problem asks us to prove that $AZ = ZX$.



Since $\angle BPC = \angle BQC = 90^\circ$ we conclude that $BPQC$ is a cyclic quadrilateral. Hence $BPYQC$ is a cyclic pentagon. A quick angle chase gives us:

$$\begin{aligned}
\angle ZPA &= 180^\circ - \angle BPY && \text{(supplimentary)}\\
&= \angle BCY && (BPCY \text{ is cyclic})\\
&= 180^\circ - \angle XCY && \text{(supplimentary)}\\
&= \angle XAY && (XACY \text{ is cyclic})\\
&= \angle ZAY.
\end{aligned}$$

Hence triangles $\triangle ZPA$ and $\triangle ZAY$ are similar ($\angle Z$ is shared). Therefore $\frac{ZP}{ZA} = \frac{ZA}{ZY}$, and hence $ZA^2 = ZP \times ZY$. Another angle chase gives us:

$$\begin{aligned}
\angle XPZ &= \angle QPY\\
&= \angle QCY && (BPCY \text{ is cyclic})\\
&= \angle ACY\\
&= \angle AXY && (ACYX \text{ is cyclic})\\
&= \angle ZXY.
\end{aligned}$$

Hence triangles $\triangle ZPX$ and $\triangle ZXY$ are similar ($\angle Z$ is shared). Therefore $\frac{ZP}{ZX} = \frac{ZX}{ZY}$ and hence $ZX^2 = ZP \times ZY$. Putting this together gives us:

$$ZA^2 = ZP \times ZY = ZX^2.$$

Hence $ZA = ZX$, and therefore Z is the midpoint of AX.

Alternative Solution (outline):
First we will embed the diagram in the Argand plane, such that point B is represented by the complex number $b = -1$ and point C is represented by the complex number $c = 1$. Lower case letters will always denote the complex number representing the corresponding upper case letter (so a is the complex number representing point A and x is the complex number representing point X, etc). We will endeavour to find expressions for all the points in the diagram in terms of p and q.

Since $\angle BPC = \angle BQC = 90^\circ$, we know that points P and Q both lie on the unit circle. So $p\overline{p} = q\overline{q} = 1$. Therefore the circumcircle of triangle PQC is the unit circle and thus $y\overline{y} = 1$ too. Since point A is the intersection of chords BP and CQ, we can compute a using the formula for the intersection of two chords.

$$a = \frac{cq(b+p) - bp(c+q)}{cq - bp} = \frac{q(-1+p) - (-1)p(1+q)}{q - (-1)p} = \frac{2pq + p - q}{p+q}$$

$$\implies \qquad \overline{a} = \frac{2\overline{pq} + \overline{p} - \overline{q}}{\overline{p} + \overline{q}} = \frac{(2\overline{pq} + \overline{p} - \overline{q})pq}{(\overline{p} + \overline{q})pq} = \frac{2 + q - p}{q + p}$$

$$\text{note:} \qquad \frac{1-a}{1-\overline{a}} = \frac{1 - \frac{2pq+p-q}{p+q}}{1 - \frac{2+q-p}{q+p}} = \frac{(p+q) - (2pq + p - q)}{(p+q) - (2+q-p)} = \frac{2q - 2pq}{2p - 2} = -q.$$

Similarly X is the intersection of chords PQ and BC, so

$$x = \frac{pq(b+c) - bc(p+q)}{pq - bc} = \frac{pq(0) - (-1)(p+q)}{pq - (-1)} = \frac{p+q}{pq+1}$$

Now we have formulas for a and x in terms of p and q. Next we will use the fact that $AYCX$ is cyclic to find a formula for y in terms of p and q. $AYCX$ being cyclic is equivalent to $\angle CAY = \angle CXY$. This is equivalent to

$$\left(\frac{c-a}{\overline{c}-\overline{a}}\right) \Big/ \left(\frac{y-a}{\overline{y}-\overline{a}}\right) = \left(\frac{c-x}{\overline{c}-\overline{x}}\right) \Big/ \left(\frac{y-x}{\overline{y}-\overline{x}}\right)$$

To simplify this, first recall that $\frac{c-a}{\overline{c}-\overline{a}} = \frac{1-a}{1-\overline{a}} = -q$. Also, since $c = \overline{c}$ and $x = \overline{x}$ (c and x are real numbers) the $\frac{c-x}{\overline{c}-\overline{x}}$ factor is 1. Furthermore, since $y\overline{y} = 1$ we can replace $\overline{y}$ with y^{-1}. Thus the equation for $AYCX$ being cyclic becomes:

$$(-q) \Big/ \left(\frac{y-a}{y^{-1}-\overline{a}}\right) = 1 \Big/ \left(\frac{y-x}{y^{-1}-x}\right).$$

From here, We can multiply out the denominators, expand the brackets and collect like terms to get a quadratic in y.

$$(q\overline{a}+x)\,y^2 - (q\overline{a}x + q + xa + 1)\,y + (qx+a) = 0.$$

Now (using $\frac{1-a}{1-\overline{a}} = -q$) we can get $(q\overline{a} + x + qx + a) = (q\overline{a}x + q + xa + 1)$. Thus the quadratic factorises as:

$$(y-1)\Big((q\overline{a}+x)y - (qx+a)\Big) = 0.$$

Since Y and C are distinct points, we know $y \neq 1$ and so we finally get a formula for y

$$(q\overline{a}+x)y - (qx+a) = 0 \qquad \Longrightarrow \qquad y = \frac{qx+a}{q\overline{a}+x}$$

We can now subsitute our formulas for a and x ($a = \frac{2pq+p-q}{p+q}$ and $x = \frac{p+q}{pq+1}$) into this expression to find y in terms of p and q. After some algebraic simplification this yields:

$$y = \frac{2p^2q + (q+1)p + q^2 - q}{(1-q)p^2 + (q+q^2)p + 2q}.$$

Let M be the midpoint of AX. So

$$m = \frac{a+x}{2} = \frac{\frac{2pq+p-q}{p+q} + \frac{p+q}{pq+1}}{2} = \frac{(pq+1)(2pq+p-q)+(p+q)^2}{2(p+q)(pq+1)}.$$

$$\begin{aligned}
\overline{m} &= \frac{(pq+1)(2+q-p)+(p+q)^2}{2(p+q)(pq+1)}.\\
p\overline{m} - 1 &= \frac{(p-1)((1-q)p^2+(q+q^2)p+2q)}{2(p+q)(pq+1)}\\
y(p\overline{m}-1) &= \frac{(p-1)(2p^2q+(q+1)p+q^2-q)}{2(p+q)(pq+1)}\\
y(p\overline{m}-1)+m &= \frac{(p-1)(2p^2q+(q+1)p+q^2-q)}{2(p+q)(pq+1)} + \frac{(pq+1)(2pq+p-q)+(p+q)^2}{2(p+q)(pq+1)}\\
&= \frac{(p-1)(2p^2q+(q+1)p+q^2-q)+(pq+1)(2pq+p-q)+(p+q)^2}{2(p+q)(pq+1)}\\
&= \frac{2p^3q+2p^2q^2+2p^2+2pq}{2(p+q)(pq+1)}\\
&= p.
\end{aligned}$$

We have shown that $y(p\overline{m}-1)+m = p$. Hence

$$m = p + y - py\overline{m}.$$

Which interpeted geometrically means that point M lies on chord PY of the unit circle.

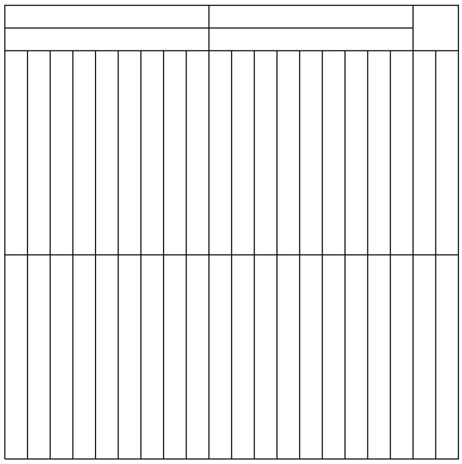
7. **Problem:** Josie and Ross are playing a game on a 20×20 chessboard. Initially the chessboard is empty. The two players alternately take turns, with Josie going first. On Josie's turn, she selects any two different empty cells, and places one white stone in each of them. On Ross' turn, he chooses any one white stone currently on the board, and replaces it with a black stone. If at any time there are 8 consecutive cells in a line (horizontally or vertically) all of which contain a white stone, Josie wins. Is it possible that Ross can stop Josie winning — regardless of how Josie plays?

   Solution: Ross can't stop Josie winning — Josie has a strategy in which she can ensure that there will be 8 white stones in a row. We will give an explicit example of such a strategy.

   To simplify notation, we define a k-strip to be a 1×8 rectangle, in which the first k cells are filled with white stones and the other $8-k$ cells are empty.

   - **Step One.** Josie creates 32 disjoint 1-strips using the following technique.

     Start by finding 44 disjoint 1×9 retangles on the board.

     – Josie places a white stone in one end of each of these 44 rectangles on her first 22 turns.

     – Ross "ruins" 22 of them. Each of the other 22 rectangles are 1-strips (by ignoring the empty end cell).

     – On Josie's next 10 turns she then chooses 20 of the ruined 1×9 rectangles, and "unruins" them by placing a white stone in the other end. These are now 1-strips (by ignoring the cell containing the black stone).

     – However Ross "ruins" a further 10 of them. So in total we have

     $$22 + 20 - 10 = 32 \text{ disjoint 1-strips.}$$

   - **Step Two.** Repeat the following operation for $k = 1, 2, 3, 4, 5$.
     Starting with 2^{6-k} disjoint k-strips. Josie can use her next 2^{5-k} turns to add one white stone to each of these strips. On Ross' next 2^{5-k} turns he can spoil at most 2^{5-k} of them. So we are left with at least 2^{5-k} disjoint $(k+1)$-strips.

   - **Step Three.** Now we have at least one 6-strip. Josie wins immediately by placing both her white stones into the empty cells in the 6-strip.

8. **Problem:** For a positive integer x, define a sequence $a_0, a_1, a_2, \ldots$ according to the following rules: $a_0 = 1$, $a_1 = x + 1$ and

$$a_{n+2} = xa_{n+1} - a_n \qquad \text{for all } n \geq 0.$$

Prove that there exist infinitely many positive integers x such that this sequence does not contain a prime number.

Solution: For each integer $n \geq 0$ and $x > 2$, we recursively define integers $a_n(x)$ and $b_n(x)$. Let $a_0(x) = b_0(x) = 1$ and $a_1(x) = x + 1$ and $b_1(x) = x - 1$. For all $n \geq 0$ let

$$a_{n+2}(x) = xa_{n+1}(x) - a_n(x) \quad \text{and} \quad b_{n+2}(x) = xb_{n+1}(x) - b_n(x).$$

So for constant x, the sequences (a_n and b_n), both satisfy the same recurrence but have different initial conditions. The chracteristic polynomial for this recurrence is $\lambda^2 - x\lambda + 1$. Now let

$$\beta = \beta(x) = \frac{x + \sqrt{x^2 - 4}}{2}$$

and note that the roots of the characteristic polynomial are β and β^{-1} (the roots are reciptricals because the constant term is 1).

Lemma:

$$a_n(x) = \frac{\beta^{n+1} - \beta^{-n}}{\beta - 1} \qquad \text{and} \qquad b_n(x) = \frac{\beta^{n+1} + \beta^{-n}}{\beta + 1}.$$

Proof of Lemma:
Since these formalas for $a_n(x)$ and $b_n(x)$ are both linear combinations of β^n and β^{-n}, we simply need to verify them for $n = 0$ and $n = 1$.

$$\frac{\beta^{0+1} - \beta^{-0}}{\beta - 1} = 1 = a_0(x) \qquad \text{and} \qquad \frac{\beta^{0+1} + \beta^{-0}}{\beta + 1} = 1 = b_0(x)$$

Since β is a root of $\lambda^2 - x\lambda + 1$, this means that $\beta^2 + 1 = x\beta$ and thus $\beta + \beta^{-1} = x$. So

$$\begin{aligned} \frac{\beta^2 - \beta^{-1}}{\beta - 1} &= \beta + 1 + \beta^{-1} \\ &= x + 1 \\ &= a_1(x). \end{aligned} \qquad\qquad \begin{aligned} \frac{\beta^2 + \beta^{-1}}{\beta + 1} &= \beta - 1 + \beta^{-1} \\ &= x - 1 \\ &= b_1(x). \end{aligned}$$

□

Thus the Lemma is now proven. Also note that $(\beta(x))^2 = \beta(y)$ for $y = x^2 - 2$, because

$$(\beta(x))^2 = \left(\frac{x + \sqrt{x^2 - 4}}{2}\right)^2 = \frac{(x^2 - 2) + \sqrt{(x^2 - 2)^2 - 4}}{2} = \frac{y + \sqrt{y^2 - 4}}{2} = \beta(y).$$

Now we use $\beta^2 = (\beta(x))^2 = \beta(y)$ to show that $a_n(x)b_n(x) = a_n(y)$.

$$\begin{aligned} a_n(x)b_n(x) &= \left(\frac{\beta^{n+1} - \beta^{-n}}{\beta - 1}\right)\left(\frac{\beta^{n+1} + \beta^{-n}}{\beta + 1}\right) \\ &= \frac{(\beta^2)^{n+1} - (\beta^2)^{-n}}{(\beta^2 - 1)} \\ &= \frac{(\beta(y))^{n+1} - (\beta(y))^{-n}}{(\beta(y) - 1)} \\ &= a_n(y). \end{aligned}$$

This shows that the sequence $a_1(y), a_2(y), a_3(y), \ldots$ cannot contain any prime numbers. Since there are infinitely many integers of the form $y = x^2 - 2$, we are done.

Alternative Solution: First note that (for $x \geq 3$) each term in the sequence is more than double the previous term, because

$$a_{n+1} = xa_n - a_{n-1} = (x-1)a_n + (a_n - a_{n-1}) \geq (x-1)a_n \geq 2a_n.$$

Moreover we can easily verify that $a_n > x$ for all $n \geq 1$ because $a_n \geq a_1 = x + 1$. Now we prove the following Lemma.

Lemma:

$$a_n^2 - a_{n+1}a_{n-1} = x + 2$$

Proof of Lemma: (by Induction)
First note that $a_2 = xa_1 - a_0 = x(x+1) - 1$. Now for the base case, we check this expression for $n = 1$.

$$a_1^2 - a_2a_0 = (x+1)^2 - (x(x+1) - 1)(1) = x + 2.$$

This proves the Lemma when $n = 0$. Now for the inductive step, we assume the Lemma is true when $n = k$. Assume: $a_k^2 - a_{k+1}a_{k-1} = x + 2$. Now we will use $xa_k = a_{k+1} + a_{k-1}$ to simplify $a_{k+1}^2 - a_{k+2}a_k$.

$$\begin{aligned} a_{k+1}^2 - a_{k+2}a_k &= a_{k+1}^2 - (xa_{k+1} - a_k)a_k \\ &= a_{k+1}^2 - xa_ka_{k+1} + a_k^2 \\ &= a_{k+1}^2 - (a_{k+1} + a_{k-1})a_{k+1} + a_k^2 \\ &= a_{k+1}^2 - a_{k+1}^2 - a_{k-1}a_{k+1} + a_k^2 \\ &= a_k^2 - a_{k-1}a_{k+1} \\ &= x + 2. \end{aligned}$$

□

Thus that the Lemma is proven. Now lets assume that $(x+2)$ is a perfect square; let $x + 2 = c^2$ for some integer $c \geq 3$. This implies that $a_n^2 - a_{n+1}a_{n-1} = c^2$. Therefore

$$a_{n+1}a_{n-1} = a_n^2 - c^2 = (a_n + c)(a_n - c)$$

and thus $(a_n + c)(a_n - c)$ is a multiple of a_{n+1}. We know that both factors $(a_n + c)$ and $(a_n - c)$ are positive because $a_n > x = c^2 - 2 > c$ for all $c \geq 3$.

Now suppose for the sake of contradiction that a_{n+1} was prime. This would mean that either $(a_n + c)$ or $(a_n - c)$ is a multiple of a_{n+1}. Therefore either

$$a_{n+1} \leq a_n + c \qquad \text{or} \qquad a_{n+1} \leq a_n - c.$$

Either way we get $a_{n+1} \leq a_n + c$. Now using $a_n > c$ we obtain

$$a_{n+1} \leq a_n + c < a_n + a_n = 2a_n.$$

However this contradicts the fact that each term in the sequence is double the previous term. Therefore a_{n+1} cannot be prime (when $x = c^2 - 2$) for any $n \geq 1$. Finally we notice that $a_0 = 1$ is not prime, and $a_1 = x + 1 = (c^2 - 2) + 1 = (c+1)(c-1)$ is clearly composite for all $c \geq 3$. Thus we have shown that the the sequence $a_0, a_1, a_2, \ldots$ contains no prime numbers whenever x is in the form $x = c^2 - 2$ for any integer $c \geq 3$.