New Zealand Mathematical Olympiad Committee

NZMO Round One 2019 — Solutions

1. *How many positive integers less than 2019 are divisible by either 18 or 21, but not both?*

   Solution: For any positive integer n, the number of multiples of n less than or equal to 2019 is given by
   $$\left\lfloor \frac{2019}{n} \right\rfloor.$$
   So there are $\left\lfloor \frac{2019}{18} \right\rfloor = 112$ multiples of 18, and $\left\lfloor \frac{2019}{21} \right\rfloor = 96$ multiples of 21. Moreover, since $lcm(18, 21) = 126$ there are $\left\lfloor \frac{2019}{126} \right\rfloor = 16$ positive integers less than 2019 which are a multiple of both 18 and 21. Therefore the final answer is
   $$\left\lfloor \frac{2019}{18} \right\rfloor + \left\lfloor \frac{2019}{21} \right\rfloor - 2\left\lfloor \frac{2019}{126} \right\rfloor = 112 + 96 - 2 \times 16 = 176.$$
   □

2. *Find all real solutions to the equation*
   $$(x^2 + 3x + 1)^{x^2 - x - 6} = 1.$$

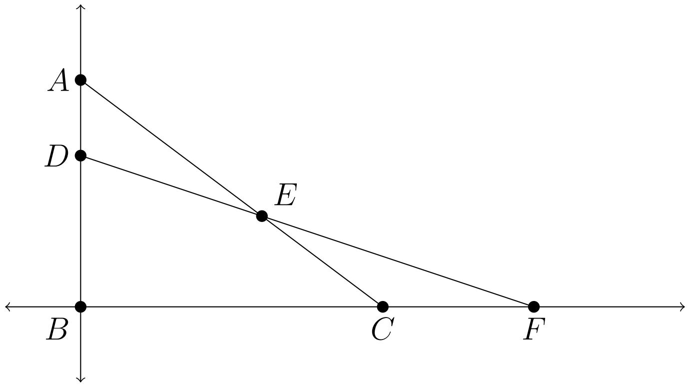
   Solution: Let $a = x^2 + 3x + 1$ and let $b = x^2 - x - 6$. The only way to have $a^b = 1$, is if $a = \pm 1$ or $b = 0$.

   - If $b = 0$, then we solve the quadratic $x^2 - x - 6 = 0$ which has solutions $x = -2, 3$. (we would also have to check that $a \neq 0$ in this case)
   - If $a = 1$, then we solve the quadratic $x^2 + 3x + 1 = 1$ which has solutions $x = 0, -3$.
   - If $a = -1$, then we solve the quadratic $x^2 + 3x + 1 = -1$ which has solutions $x = -1, -2$. (we also have to check that b is an even integer in this case)

   Therefore there are a total of 5 candidate solutions: $x = -3, -2, -1, 0, 3$.
   Remark: In order to receive full marks, a student would have to demonstrate that $x = -3, -2, -1, 0, 3$ are actually all solutions, by substituting each of these values into the expression, and verify that the result is indeed 1. □

3. *In triangle ABC, points D and E lie on the interior of segments AB and AC, respectively, such that $AD = 1$, $DB = 2$, $BC = 4$, $CE = 2$ and $EA = 3$. Let DE intersect BC at F. Determine the length of CF.*

   Solution: First notice that the sidelengths of $\triangle ABC$ are 3, 4 and 5. By Pythagoras this implies that triangle ABC is right-angled at B. Now we can put the diagram on coordinate axes such that $B = (0, 0)$ and $A = (0, 3)$ and $C = (4, 0)$. Furthermore we get $D = (0, 2)$ and since E divides CA into the ratio $2 : 3$ we get $E = (2.4, 1.2)$, as shown in the diagram.



Now we can calculate the slope of the line DE to be $\frac{-0.8}{2.4} = -\frac{1}{3}$. This means that the equation of line DE is given by $y = -\frac{x}{3} + 2$. Therefore the x-intercept of this line is the solution to $0 = -\frac{x}{3} + 2$. The solution is when $x = 6$, and thus $F = (6, 0)$. Hence $CF = 2$. □

4. *Show that the number* $122^n - 102^n - 21^n$ *is always one less than a multiple of* 2020, *for any positive integer* n.

Solution: Let $f(n) = 122^n - 102^n - 21^n$. We consider $f(n)$ in mod 101 and in mod 20 separately.

- Consider $f(n)$ mod 101.

$$\begin{aligned} f(n) &= 122^n - 102^n - 21^n \\ &\equiv 21^n - 1^n - 21^n \pmod{101} \\ &= -1 \end{aligned}$$

- Consider $f(n)$ mod 20.

$$\begin{aligned} f(n) &= 122^n - 102^n - 21^n \\ &\equiv 2^n - 2^n - 1^n \pmod{20} \\ &= -1 \end{aligned}$$

Therefore $f(n) \equiv -1$ both in mod 20 and in mod 101. Since 20 and 101 are relatively prime, this means $f(n) \equiv -1 \pmod{2020}$. As required. □

5. *Find all positive integers* n *such that* $n^4 - n^3 + 3n^2 + 5$ *is a perfect square.*

Solution: Let $f(n) = 4n^4 - 4n^3 + 12n^2 + 20 = 4(n^4 - n^3 + 3n^2 + 5)$ and note that $(n^4 - n^3 + 3n^2 + 5)$ is a perfect square if and only if $f(n)$ is. First note that:

$$(2n^2 - n + 5)^2 - f(n) = 9n^2 - 10n + 5 = 4n^2 + 5(n-1)^2 > 0.$$

Also note that

$$f(n) - (2n^2 - n + 2)^2 = 3n^2 + 4n + 16 = 2n^2 + (n+2)^2 + 12 > 0.$$

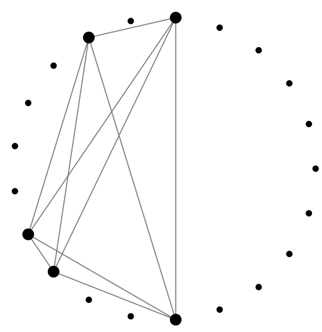
Therefore $(2n^2 - n + 2)^2 < f(n) < (2n^2 - n + 5)^2$, so the only way $f(n)$ could be a perfect square is if it is $(2n^2 - n + 3)^2$ or $(2n^2 - n + 4)^2$. Solving $f(n) = (2n^2 - n + 3)^2$ gives us the quadratic $n^2 - 6n - 11 = 0$ which has no integer solutions. Solving $f(n) = (2n^2 - n + 4)^2$ gives us $5n^2 - 8n - 4 = (5n + 2)(n - 2) = 0$. which has only one integer solution $n = 2$. Checking

$$(2)^4 - (2)^3 + 3(2)^2 + 5 = 25$$

which is a perfect square. Therefore the only solution is $n = 2$. □

6. *Let V be the set of vertices of a regular 21-gon. Given a non-empty subset U of V, let $m(U)$ be the number of distinct lengths that occur between two distinct vertices in U. What is the maximum value of $\frac{m(U)}{|U|}$ as U varies over all non-empty subsets of V?*

Solution: To simplify notation, we will let m be $m(U)$ and let n be $|U|$. First note that there are 10 different diagonal-lengths in a regular 21-gon. Now consider the following set of 5 vertices.



Note that each of the 10 different diagonal-lengths appear (exactly once each). So for this set of 5 vertices we have $\frac{m}{n} = \frac{10}{5} = 2$. We will now show that this is the maximum possible value for $\frac{m}{n}$. If U is an arbitrary non-empty set of vertices, then there are two cases:

- Case 1: $n < 5$. The total number of pairs of vertices in U is given by $\frac{1}{2}n(n-1)$. Since $n - 1 < 4$ this gives us the bound:

$$m \leq \frac{n(n-1)}{2} < \frac{n \times 4}{2} = 2n.$$

  Thus $\frac{m}{n} < 2$ in this case.

- Case 1: $n \geq 5$. The total number of distances in U is at most 10 because there are only 10 different diagonal lengths in the 21-gon. Therefore

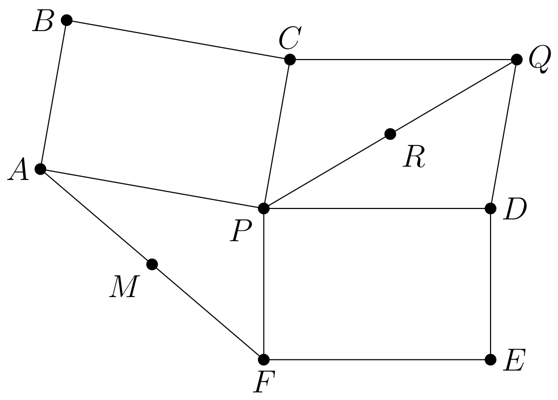
$$\frac{m}{n} \leq \frac{10}{n} \leq \frac{10}{5} = 2$$

  as required.

Remark: The construction given is unqiue up to rotations and reflections. *I.e.* all sets that achieve the value $\frac{m}{n} = 2$ are congruent to the example given here. □

7. *Let $ABCDEF$ be a convex hexagon containing a point P in its interior such that $PABC$ and $PDEF$ are congruent rectangles with $PA = BC = PD = EF$ (and $AB = PC = DE = PF$). Let ℓ be the line through the midpoint of AF and the circumcentre of PCD. Prove that ℓ passes through P.*

Solution: Let M be the midpoint of AF and let O be the circumcentre of triangle CPD. Now construct Q to be the point such that $CPDQ$ is a parallelogram, and let R be the centre of this parallelogram (*i.e.* R is the intersection of PQ with CD, and also R is the midpoint of PQ).



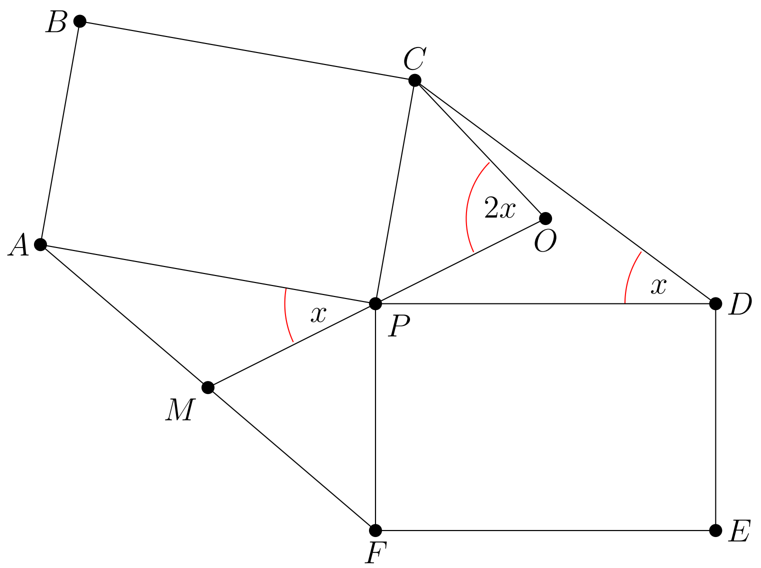
Note that $QD = CP = FP$ and $DP = PA$ and $\angle QDP = 180° - \angle DPC = \angle FPA$. Therefore (by SAS) we have a pair of congruent triangles:

$$\triangle QDP \cong \triangle FPA.$$

Therefore $\angle MAP = \angle RPD$ and $AF = PQ$. Thus $AM = \frac{1}{2}AF = \frac{1}{2}PQ = PR$. Therefore (by SAS) we have another pair of congruent triangles:

$$\triangle MAP \cong \triangle RPD.$$

Therefore $\angle APM = \angle RDP$. Let $x = \angle APM$ so that $\angle CDP = \angle RDP = x$ also.



Since the angle subtended at the circumcentre is double the angle subtended at the circumference, we get $\angle COP = 2x$ (recall that O is the circumcentre of $\triangle PCD$). Finally we get $\angle OPC = 90° - x$ because $\triangle COP$ is isosceles. Putting this all together, we get

$$\angle OPM = \angle OPC + \angle CPA + \angle APM = (90° - x) + 90° + x = 180°.$$

Therefore $\angle OPM$ is a straight line. □

8. *Suppose that* $x_1, x_2, x_3, \dots x_n$ *are real numbers between* 0 *and* 1 *with sum* s. *Prove that*

$$\sum_{i=1}^{n} \frac{x_i}{s+1-x_i} + \prod_{i=1}^{n}(1-x_i) \leq 1.$$

Solution: Let i be arbitrary and consider the set $A = \{a_1, a_2, \dots, a_n\}$ defined by $a_i = s + 1 - x_i$ and let $a_j = 1 - x_j$ for all $j \neq i$. For example, if $i = 2$ then A would be $\{1 - x_1, s + 1 - x_2, 1 - x_3, \dots, 1 - x_n\}$. The AM-GM inequality on A tells us

$$1 = \frac{(s+1-x_i) + \sum_{j\neq i}(1-x_j)}{n} \geq \left((1+s-x_i)\prod_{j\neq i}(1-x_j)\right)^{\frac{1}{n}}$$

Which rearranges to give us

$$1 - (s+1-x_i)\prod_{j\neq i}(1-x_j) \geq 0.$$

From here we can multiply both sides by $(1 - x_i)$, then add s to both sides and factorise the LHS to get:

$$(s+1-x_i)\left(1 - \prod_{j=1}^{n}(1-x_j)\right) \geq s.$$

Now multipy both sides by $\frac{x_i}{s(s+1-x_i)}$ to get the following equation.

$$\left(1 - \prod_{j=1}^{n}(1-x_j)\right)\frac{x_i}{s} \geq \frac{x_i}{s+1-x_i} \qquad (1)$$

Note that this equation holds for all i. Now consider the sum of Equation 1 over all $1 \leq i \leq n$. Since $(1 - \prod(1 - x_j))$ is constant and $\sum \frac{x_i}{s} = 1$, the sum of all the LHS equals $\left(1 - \prod(1 - x_j)\right)$. So we get

$$1 - \prod_{j=1}^{n}(1-x_j) \geq \sum_{i=1}^{n}\frac{x_i}{s+1-x_i}$$

as required. □