New Zealand Mathematical Olympiad Committee

NZMO Round One 2019

1. How many positive integers less than 2019 are divisible by either 18 or 21, but not both?

2. Find all real solutions to the equation
$$(x^2 + 3x + 1)^{x^2 - x - 6} = 1.$$

3. In triangle ABC, points D and E lie on the interior of segments AB and AC, respectively, such that $AD = 1$, $DB = 2$, $BC = 4$, $CE = 2$ and $EA = 3$. Let DE intersect BC at F. Determine the length of CF.

4. Show that the number $122^n - 102^n - 21^n$ is always one less than a multiple of 2020, for any positive integer n.

5. Find all positive integers n such that $n^4 - n^3 + 3n^2 + 5$ is a perfect square.

6. Let V be the set of vertices of a regular 21-gon. Given a non-empty subset U of V, let $m(U)$ be the number of distinct lengths that occur between two distinct vertices in U. What is the maximum value of $\frac{m(U)}{|U|}$ as U varies over all non-empty subsets of V?

7. Let $ABCDEF$ be a convex hexagon containing a point P in its interior such that $PABC$ and $PDEF$ are congruent rectangles with $PA = BC = PD = EF$ (and $AB = PC = DE = PF$). Let ℓ be the line through the midpoint of AF and the circumcentre of PCD. Prove that ℓ passes through P.

8. Suppose that $x_1, x_2, x_3, \ldots x_n$ are real numbers between 0 and 1 with sum s. Prove that
$$\sum_{i=1}^{n} \frac{x_i}{s + 1 - x_i} + \prod_{i=1}^{n}(1 - x_i) \leq 1.$$

July 2019

www.mathsolympiad.org.nz