New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2018

Due: 28th September 2018

1. Suppose that a, b, c and d are four different integers. Explain why

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

must be a multiple of 12.

2. Find all pairs of integers (a, b) such that

$$a^2 + ab - b = 2018.$$

3. Show that amongst any 8 points in the interior of a 7×12 rectangle, there exists a pair whose distance is less than 5.

Note: The interior *of a rectangle excludes points lying on the sides of the rectangle.*

4. Let P be a point inside triangle ABC such that $\angle CPA = 90°$ and $\angle CBP = \angle CAP$. Prove that $\angle PXY = 90°$, where X and Y are the midpoints of AB and AC respectively.

5. Let a, b and c be positive real numbers satisfying

$$\frac{1}{a+2019} + \frac{1}{b+2019} + \frac{1}{c+2019} = \frac{1}{2019}.$$

Prove that $abc \geq 4038^3$.

6. The intersection of a cube and a plane is a pentagon. Prove the length of at least one side of the pentagon differs from 1 metre by at least 20 centimetres.

7. Let N be the number of ways to colour each cell in a 2×50 rectangle either red or blue such that each 2×2 block contains at least one blue cell. Show that N is a multiple of 3^{25}, but not a multiple of 3^{26}.

8. Let λ be a line and let M, N be two points on λ. Circles α and β centred at A and B respectively are both tangent to λ at M, with A and B being on opposite sides of λ. Circles γ and δ centred at C and D respectively are both tangent to λ at N, with C and D being on opposite sides of λ. Moreover A and C are on the same side of λ. Prove that if there exists a circle tangent to all circles $\alpha, \beta, \gamma, \delta$ containing all of them in its interior, then the lines AC, BD and λ are either concurrent or parallel.

9. Let x, y, p, n, k be positive integers such that

$$x^n + y^n = p^k.$$

Prove that if $n > 1$ is odd, and p is an odd prime, then n is a power of p.

10. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x)f(y) = f(xy+1) + f(x-y) - 2$$

for all $x, y \in \mathbb{R}$.

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