New Zealand Mathematical Olympiad Committee

NZMO Round Two 2024 — Instructions

Exam date: 30^{th} August

General Instructions

1. You have 3 hours to work on the exam.
2. There are 5 problems, each worth equal marks. You should attempt all 5 problems. You may work on them in any order.
3. Geometrical instruments (ruler and compasses) may be used. Calculators, phones, computers and electronic devices of any sort are not permitted.
4. Write your solutions on the paper provided. Let the supervisor know if you need more. Write your name at the top of every page as you start using it.
5. Full written solutions — not just answers — are required, with complete proofs of any assertions you make. Your marks will depend on the clarity of your mathematical presentation. Work in rough first, and then draft a neat final version of your best attempt.
6. Make sure you fill in your details on the Declaration Form. Hand in all of your rough work, in addition to your neat solutions. The declaration form is used as a cover sheet for your submission.
7. At the end of the exam, remain seated quietly until all scripts have been collected and the supervisor indicates that you are free to move.
8. You may not take the question paper from the exam room.
9. The contest problems are to be kept confidential until they are posted on the NZMOC webpage www.mathsolympiad.org.nz. Do not disclose or discuss the problems online until this has occurred. Typically this will be approximately one week after the exam has been sat.
10. **Do not turn over until told to do so**.

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NZMO Round Two 2024 — Problems

Exam date: 30^{th} August

- There are 5 problems. You should attempt to find solutions for as many as you can.
- Solutions (that is, answers with justifications) and not just answers are required for all problems, even if they are phrased in a way that only asks for an answer.
- Read and follow the "General instructions" accompanying these problems.

Problems

1. At each vertex of a regular 14-gon, lies a coin. Initially 7 coins are heads, and 7 coins are tails. Determine the minimum number t such that it's always possible to turn over at most t of the coins so that in the resulting 14-gon, no two adjacent coins are both heads and no two adjacent coins are both tails.
2. Consider the sequence $a_1, a_2, a_3, \ldots$ defined by $a_1 = 2024^{2024}$ and for each positive integer n,
$$a_{n+1} = \left|a_n - \sqrt{2}\right|.$$
Prove that there exists an integer k such that $a_{k+2} = a_k$.
Here $|x|$ denotes the absolute value of x.
3. Let A, B, C, D, E be five different points on the circumference of a circle in that (cyclic) order. Let F be the intersection of chords BD and CE. Show that if $AB = AE = AF$ then lines AF and CD are perpendicular.
4. Determine all positive integers n less than 2024 such that for all positive integers x, the greatest common divisor of $9x + 1$ and $nx + 1$ is 1.
5. Determine the least real number L such that
$$\frac{1}{a} + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} \leqslant L$$
for all quadruples (a, b, c, d) of integers satisfying $1 < a < b < c < d$.