New Zealand Mathematical Olympiad Committee

NZMO Round Two 2023 — Instructions
Exam date: 25th August

General Instructions

1. You have 3 hours to work on the exam.

2. There are 5 problems, each worth equal marks. You should attempt all 5 problems. You may work on them in any order.

3. Geometrical instruments (ruler and compasses) may be used. Calculators, phones, computers and electronic devices of any sort are not permitted.

4. Write your solutions on the paper provided. Let the supervisor know if you need more. Write your name at the top of every page as you start using it.

5. Full written solutions — not just answers — are required, with complete proofs of any assertions you make. Your marks will depend on the clarity of your mathematical presentation. Work in rough first, and then draft a neat final version of your best attempt.

6. Make sure you fill in your details on the cover sheet. Hand in all of your rough work, in addition to your neat solutions.

7. At the end of the exam, remain seated quietly until all scripts have been collected and the supervisor indicates that you are free to move.

8. You may not take the question paper from the exam room.

9. The contest problems are to be kept confidential until they are posted on the NZMOC webpage www.mathsolympiad.org.nz. Do not disclose or discuss the problems online until this has occurred.

10. **Do not turn over until told to do so**.

NZMO Round Two 2023 — Problems

Exam date: 25th August

- There are 5 problems. You should attempt to find solutions for as many as you can.

- Solutions (that is, answers with justifications) and not just answers are required for all problems, even if they are phrased in a way that only asks for an answer.

- Read and follow the "General instructions" accompanying these problems.

Problems

1. For any positive integer n let $n! = 1 \times 2 \times 3 \times \cdots \times n$. Do there exist infinitely many triples (p, q, r), of positive integers with $p > q > r > 1$ such that the product

$$p! \cdot q! \cdot r!$$

is a perfect square?

2. Let a, b and c be positive real numbers such that $a + b + c = abc$. Prove that at least one of a, b or c is greater than $\frac{17}{10}$.

3. Let $ABCD$ be a square (vertices labelled in clockwise order). Let Z be any point on diagonal AC between A and C such that $AZ > ZC$. Points X and Y exist such that $AXYZ$ is a square (vertices labelled in clockwise order) and point B lies inside $AXYZ$. Let M be the point of intersection of lines BX and DZ (extended if necessary). Prove that C, M and Y are colinear.

4. For any positive integer n, let $f(n)$ be the number of subsets of $\{1, 2, \ldots, n\}$ whose sum is equal to n. Does there exist infinitely many positive integers m such that $f(m) = f(m + 1)$?
(Note that each element in a subset must be distinct.)

5. Let x, y and z be real numbers such that: $x^2 = y + 2$, and $y^2 = z + 2$, and $z^2 = x + 2$.
Prove that $x + y + z$ is an integer.