New Zealand Mathematical Olympiad Committee

NZMO Round One 2023 — Instructions

Submissions due date: 30^{th} June

The **New Zealand Mathematical Olympiad** (NZMO) consists of two rounds:

Round One (the NZMO1): A take home exam (the following set of 8 problems). Solutions are to be submitted by 30^{th} *June* (21:00 NZST).

Round Two (the NZMO2): A three hour supervised exam in August.

Awards for the NZMO (gold/silver/bronze/honourable mention) will be made based on the results of the NZMO2. Scores will not be announced for either round. Participants in the NZMO1 will only receive an indication of whether they have been invited to participate in the NZMO2; participants in the NZMO2 will just be told the level of their award, if any.

In addition, the results of both rounds of the NZMO will be used to select about 25 students to participate in the training camp in January 2024, and around 20 female or non-binary students to attend the training camp in April 2024. Only students who are New Zealand citizens or permanent residents, and who will still be enrolled in intermediate or high school in 2024, are eligible for the January training camp. Only students who attend this camp are eligible for selection to represent New Zealand at the 2024 International Mathematical Olympiad, (IMO).

General instructions:

- Participation in the NZMO is open to any student enrolled in the New Zealand education system, at secondary school level or below.
- The NZMO is an individual competition. Participants must work on the problems entirely on their own, without assistance from anyone else or any kind of calculator/computer (other than purely for word processing purposes). This includes but is not limited to using calculations, graphing functions, or using tools such as geogebra.
- Electronic devices may not be used to assist in solving the problems. This includes but is not limited to calculators, computers, tablets, smart phones and smart watches.
- The internet may not be used, except as a reference for looking up definitions.
- Although some problems may seem to require only a numerical answer, in order to receive full credit for the problem a complete justification must be provided.
- Submitting rough working and/or partial solutions cannot negatively affect the score awarded. We encourage you to submit any rough work or partial solutions that you think may be of value.
- It is forbidden to discuss the problems especially online until the official solutions are posted on the NZMOC website. Typically this will be 3 days after the submission deadline.
- Please address **all** queries to Dr Ross Atkins, info@mathsolympiad.org.nz

The NZMO1 submission url is: https://www.mathsolympiad.org.nz/nzmo1_submission. If you are having trouble loading the submission form, please try logging out of your google account, or opening the form in an incognito window. All your solutions and partial solutions should be submitted as a **single document** in PDF format. Please complete the submission form **carefully**, especially your contact details. If you have any difficulties with the submission form, contact Dr Ross Atkins (info@mathsolympiad.org.nz).

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NZMO Round One 2023 — Problems

Submissions due date: 30th June

- There are 8 problems.
- Read and follow the "General instructions" accompanying these problems.
- If any clarification is required, please contact Dr Ross Atkins (info@mathsolympiad.org.nz).

Problems

1. There are 2023 employees in the office, each of them knowing exactly 1686 of the others. For any pair of employees they either both know each other or both don't know each other. Prove that we can find 7 employees each of them knowing all 6 others.

2. Let $ABCD$ be a parallelogram, and let P be a point on the side AB. Let the line through P parallel to BC intersect the diagonal AC at point Q. Prove that

$$|DAQ|^2 = |PAQ| \times |BCD|,$$

where $|XYZ|$ denotes the area of triangle XYZ.

3. Find the sum of the smallest and largest possible values for x which satisfy the following equation.

$$9^{x+1} + 2187 = 3^{6x-x^2}$$

4. Let p be a prime and let $f(x) = ax^2 + bx + c$ be a quadratic polynomial with integer coefficients such that $0 < a, b, c \leqslant p$. Suppose $f(x)$ is divisible by p whenever x is a positive integer. Find all possible values of $a + b + c$.

5. Find all triples (a, b, n) of positive integers such that a and b are both divisors of n, and $a + b = \frac{n}{2}$.

6. Let triangle ABC be right-angled at A. Let D be the point on AC such that BD bisects angle $\angle ABC$. Prove that $BC - BD = 2AB$ if and only if $\frac{1}{BD} - \frac{1}{BC} = \frac{1}{2AB}$.

7. Let n, m be positive integers. Let $A_1, A_2, A_3, \ldots, A_m$ be sets such that $A_i \subseteq \{1, 2, 3, \ldots, n\}$ and $|A_i| = 3$ for all i (*i.e.* A_i consists of three different positive integers each at most n). Suppose for all $i < j$ we have

$$|A_i \cap A_j| \leqslant 1$$

(*i.e.* A_i and A_j have at most one element in common).

   (a) Prove that $m \leqslant \frac{n(n-1)}{6}$.

   (b) Show that for all $n \geq 3$ it is possible to have $m \geqslant \frac{(n-1)(n-2)}{6}$.

8. Find all non-zero real numbers a, b, c such that the following polynomial has four (not necessarily distinct) positive real roots.

$$P(x) = ax^4 - 8ax^3 + bx^2 - 32cx + 16c$$