New Zealand Mathematical Olympiad Committee

NZMO Round One 2020 — Instructions

Submissions due date: 28th July

The **New Zealand Mathematical Olympiad** (NZMO) consists of two rounds:

Round One (the NZMO1): A take home exam (the following set of 8 problems). Solutions are to be submitted by 28th *July*. Participation in the NZMO1 is open to any New Zealand intermediate or secondary student.

Round Two (the NZMO2): A three hour exam in September. Participation in the NZMO2 is by invitation only. Invitations will be made based on the results of the NZMO1.

Awards for the NZMO (gold/silver/bronze/honourable mention) will be made based on the results of the NZMO2. Scores will not be announced for either round. Participants in the NZMO1 will only receive an indication of whether they have been invited to participate in the NZMO2; participants in the NZMO2 will just be told the level of their award, if any.

In addition, the results of both rounds of the NZMO will be used to select about 25 students to participate in the NZMOC training camp, to be held at the University of Auckland in January 2021. Only students who are New Zealand citizens or permanent residents, and who will still be enrolled in intermediate or secondary school in 2021, are eligible for NZMOC training camp selection. The training camp is the first step in the selection process to choose the team of six students to represent New Zealand at the 2021 International Mathematical Olympiad, (IMO). Only students who attend this camp are eligible for team selection.

General instructions:

- All solutions must be entirely your own work.
- You may not use a calculator, a computer or the internet (except as a reference, for e.g. definitions) to assist you in solving the problems.
- Although some problems may seem to require only a numerical answer, in order to receive full credit for the problem a complete justification must be provided. In fact, an answer alone will be worth at most 20% of the credit for a problem.
- We do not expect many, if any, perfect submissions. So, please submit all the solutions and partial solutions that you can find.
- Please address **all** queries about the problems to Dr Ross Atkins, info.nzmoc@gmail.com

Any New Zealand intermediate or secondary student can participate. Students submitting solutions must be currently enrolled in the New Zealand education system as an intermediate or secondary student. If you are uncertain about your eligibility please contact Dr Ross Atkins (info.nzmoc@gmail.com) **before** submitting solutions.

The NZMO1 submission url is: https://www.mathsolympiad.org.nz/nzmo1_submission. All your solutions and partial solutions should be submitted as a **single document** in PDF format. Please complete the submission form **carefully**, especially your contact details. If you have any difficulties with the submission form, contact Dr Ross Atkins (info.nzmoc@gmail.com).

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NZMO Round One 2020 — Problems

Submissions due date: 28th July

- There are 8 problems. You should attempt to find solutions for as many as you can. Solutions (that is, answers with justifications) and not just answers are required for all problems, even if they are phrased in a way that only asks for an answer.
- Read and follow the "General instructions" accompanying these problems.
- If any clarification is required, please contact Dr Ross Atkins (info.nzmoc@gmail.com).

Problems

1. What is the maximum integer n such that $\dfrac{50!}{2^n}$ is an integer?
2. Let $ABCD$ be a square and let X be any point on side BC between B and C. Let Y be the point on line CD such that $BX = YD$ and D is between C and Y. Prove that the midpoint of XY lies on diagonal BD.
3. You have an unlimited supply of square tiles with side length 1 and equilateral triangle tiles with side length 1. For which n can you use these tiles to create a convex n-sided polygon? The tiles must fit together without gaps and may not overlap.
4. Determine all prime numbers p such that $p^2 - 6$ and $p^2 + 6$ are both prime numbers.
5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x + f(y)) = 2x + 2f(y + 1)$$

   for all real numbers x and y.
6. Let $\triangle ABC$ be an acute triangle with $AB > AC$. Let P be the foot of the altitude from C to AB and let Q be the foot of the altitude from B to AC. Let X be the intersection of PQ and BC. Let the intersection of the circumcircles of triangle $\triangle AXC$ and triangle $\triangle PQC$ be distinct points: C and Y. Prove that PY bisects AX.
7. Josie and Ross are playing a game on a 20×20 chessboard. Initially the chessboard is empty. The two players alternately take turns, with Josie going first. On Josie's turn, she selects any two different empty cells, and places one white stone in each of them. On Ross' turn, he chooses any one white stone currently on the board, and replaces it with a black stone. If at any time there are 8 consecutive cells in a line (horizontally or vertically) all of which contain a white stone, Josie wins. Is it possible that Ross can stop Josie winning — regardless of how Josie plays?
8. For a positive integer x, define a sequence $a_0, a_1, a_2, \ldots$ according to the following rules: $a_0 = 1$, $a_1 = x + 1$ and

$$a_{n+2} = xa_{n+1} - a_n \qquad \text{for all } n \geq 0.$$

   Prove that there exist infinitely many positive integers x such that this sequence does not contain a prime number.