New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2014 — Instructions

Solutions due date: 24th September 2014

These problems will be used by the NZMOC to select students for its International Mathematical Olympiad Training Camp, to be held in Auckland between the 11th and 17th of January 2015. Only students who attend this camp are eligible for selection to represent New Zealand at the 2015 International Mathematical Olympiad (IMO), to be held in Thailand in July 2015. We have been fortunate in obtaining some sponsorship for the camp so have been able to keep the cost to $600. This covers all expenses for the camp including travel for those from outside of Auckland. Of course, any donations in excess of this amount will be gratefully accepted and receipts will be provided for tax purposes.

At the camp a squad of 10–12 students will be chosen for further training, and to take part in several international competitions, including the Australian and Asia-Pacific Mathematical Olympiads. The New Zealand team for the 2015 IMO will be chosen from this squad.

Students will be selected into two groups for the camp: juniors and seniors.

- If you are currently in year 12, **or** you have been a member of the NZIMO training squad, then you will be considered only as a senior.
- Otherwise, you will be considered as either a junior or a senior.
- Since the problems and the participants vary from year to year it is hard to be precise about the selection criteria. However, as a rough guide, if you solve five or more of the problems completely then you will be in the running for selection as a junior. Obviously, the criteria for senior selection are somewhat higher.

General instructions:

- Although some problems seem to require only a numerical answer, in order to receive full credit for the problem a complete justification must be provided. In fact, an answer alone will only be worth 1 point out of 7.
- You may not use a calculator, computer or the internet (except as a reference for e.g. definitions) to assist you in solving the problems.
- All solutions must be entirely your own work.
- We do not expect many, if any, perfect submissions. So, please submit all the solutions and partial solutions that you can find.

Students submitting solutions should be intending to remain in school in 2015 and should also hold New Zealand Passports or have New Zealand Resident status. To be eligible for the 2015 IMO you must have been born on or after 1 July 1995, and must not be formally enrolled in a University or similar institution prior to the IMO.

Your solutions, together with a completed Registration Form (overleaf), should be sent to

> NZ Mathematics Olympiads, The University of Auckland, Department of Mathematics, Private Bag 92019, Auckland 1142, New Zealand

arriving no later than 24th September 2014. We regret that we are unable to accept electronic submissions. You will be notified whether or not you have been selected for the Camp by 22nd October 2013.

July 2014

www.mathsolympiad.org.nz

Registration Form
NZMOC Camp Selection Problems 2014

Name: ____________________

Gender: male/female

School year level in 2015: ____________________

Home address: ____________________

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Email address: ____________________

Home phone number: ____________________

School: ____________________

School address: ____________________

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Principal: ____________________

HOD Mathematics: ____________________

Do you intend to take part in the camp selection problems for any other Olympiad camp? yes/no

If so, and if selected, which camp would you prefer to attend? ____________________

Have you put your name forward for a Science camp or any other camp in January? yes/no

Are there any criminal charges, or pending criminal charges against you? yes/no

Some conditions are attached to camp selection. You must be:

- Born on or after 1 July 1995
- Studying in 2015 at a recognised secondary school in NZ
- Available in July 2015 to represent NZ overseas as part of the NZIMO team if selected.
- A NZ citizen or hold NZ resident status.

Declaration: I satisfy these requirements, have worked on the questions without assistance from anyone else, and have read, understood and followed the instructions for the January camp selection problems. I agree to being contacted through the email address I have supplied.

Signature: ____________________ Date: ____________________

Attach this registration form to your solutions, and send them to

> NZ Mathematics Olympiads, The University of Auckland, Department of Mathematics, Private Bag 92019, Auckland 1142, New Zealand,

arriving no later than 24th September 2014.

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Note: If any clarification is required, please contact Michael Albert (michael.albert@cs.otago.ac.nz).

1. Prove that for all positive real numbers a and b:

$$\frac{(a+b)^3}{4} \geq a^2b + ab^2.$$

2. Let ABC be a triangle in which the length of side AB is 4 units, and that of BC is 2 units. Let D be the point on AB at distance 3 units from A. Prove that the line perpendicular to AB through D, the angle bisector of $\angle ABC$, and the perpendicular bisector of BC all meet at a single point.

3. Find all pairs (x, y) of positive integers such that $(x+y)(x^2+9y)$ is the cube of a prime number.

4. Given 2014 points in the plane, no three of which are collinear, what is the minimum number of line segments that can be drawn connecting pairs of points in such a way that adding a single additional line segment of the same sort will always produce a triangle of three connected points?

5. Let ABC be an acute angled triangle. Let the altitude from C to AB meet AB at C' and have midpoint M, and let the altitude from B to AC meet AC at B' and have midpoint N. Let P be the point of intersection of AM and BB' and Q the point of intersection of AN and CC'. Prove that the point M, N, P and Q lie on a circle.

6. Determine all triples of positive integers a, b and c such that their least common multiple is equal to their sum.

7. Determine all pairs of real numbers (k, d) such that the system of equations:

$$\begin{aligned} x^3 + y^3 &= 2 \\ kx + d &= y, \end{aligned}$$

has no solutions (x, y) with x and y real numbers.

8. Michael wants to arrange a doubles tennis tournament among his friends. However, he has some peculiar conditions: the total number of matches should equal the total number of players, and every pair of friends should play as either teammates or opponents in at least one match. The number of players in a single match is four. What is the largest number of people who can take part in such a tournament?

9. Let AB be a line segment with midpoint I. A circle, centred at I has diameter less than the length of the segment. A triangle ABC is tangent to the circle on sides AC and BC. On AC a point X is given, and on BC a point Y is given such that XY is also tangent to the circle (in particular X lies between the point of tangency of the circle with AC and C, and similarly Y lies between the point of tangency of the circle with BC and C. Prove that $AX \cdot BY = AI \cdot BI$.

10. In the land of Microbablia the alphabet has only two letters, 'A' and 'B'. Not surprisingly, the inhabitants are obsessed with the band ABBA. Words in the local dialect with a high ABBA-factor are considered particularly lucky. To compute the ABBA-factor of a word you just count the number of occurrences of ABBA within the word (not necessarily consecutively). So for instance AABA has ABBA-factor 0, ABBA has ABBA-factor 1, AABBBA has ABBA-factor 6, and ABBABBA has ABBA factor 8. What is the greatest possible ABBA-factor for a 100 letter word?