New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2013 — Solutions

1. *You have a set of five weights, together with a balance that allows you to compare the weight of two things. The weights are known to be 10, 20, 30, 40 and 50 grams, but are otherwise identical except for their labels. The 10 and 50 gram weights are clearly labelled, but the labels have been erased on the remaining weights. Using the balance exactly once, is it possible to determine what one of the three unlabelled weights is? If so, explain how, and if not, explain why not.*

   Solution: It is possible. Weigh any two of the unlabelled weights against both the 10 and 50 gram weights. If they are lighter, the two weights must be 20 and 30 grams, so the unused weight is 40g. Similarly, if they balance it is 30g, and if they are heavier it is 20g. □

2. *Find all primes that can be written both as a sum and as a difference of two primes (note that 1 is* not *a prime).*

   Solution: Let p be such a prime. Clearly $p > 2$ since 2 is the smallest prime, so p is odd. So, in writing p as the sum of two primes we must have $p = r + 2$ (since the sum of two odd primes is even), and likewise in writing it as the difference of two primes, $p = s - 2$. So r, $r+2$ and $r+4$ are all prime. But, one of these three numbers is a multiple of 3 and since the only prime multiple of 3 is 3 itself it must be the case that $r = 3$, and hence $p = 5$ (we should check that this works but it does since 7 is also prime). □

3. *Prove that for any positive integer $n > 2$ we can find n distinct positive integers, the sum of whose reciprocals is equal to 1.*

   Solution: For $n = 3$ we have $1/2 + 1/3 + 1/6 = 1$. We also have the identity that for any positive integer k

   $$\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}$$

   Note that the two denominators on the right hand side are distinct, greater than the denominator on the left hand side, and the second is greater than the first. So, if we have t distinct positive integers whose sum is 1 we can find $t + 1$ such integers by taking the largest of them, say m and replacing it by the pair $m + 1$, $m(m + 1)$. Since we can do this procedure as often as we like we can find a set of distinct positive integers of any size whose reciprocals sum to 1. □

4. *Let C be a cube. By connecting the centres of the faces of C with lines we form an octahedron O. By connecting the centers of each face of O with lines we get a smaller cube C'. What is the ratio between the side length of C and the side length of C'?*

   Solution: Let s be the side length of the original cube. The distance between the centres of two adjacent faces (i.e. the side length of the octahedron) is equal to the hypotenuse of a right angled triangle with legs of length $s/2$ and hence equals $s/\sqrt{2}$. Now consider an octahedron of side length t. Think of it as being made up of two pyramids – a "northern" and a "southern" one. Concentrate on the southern pyramid. The base of this pyramid is a square of side length t. The square formed by connecting each of the centres of its faces together with the "south pole" also defines a pyramid, P, and there is a similar, larger, pyramid, $\hat{P}$ formed by connecting the bottom of each altitude of its triangular faces together with the south pole. The side length of the square base of $\hat{P}$ is $t/\sqrt{2}$. Since the centre of an equilateral triangle lies $2/3$ of the distance down its altitude, the side length of the square base of P is $2t/3\sqrt{2} = t\sqrt{2}/3$. But this is the side length, s', of C' and $t = s/\sqrt{2}$, so $s' = s/3$ i.e. the ratio of the side lengths is 3. □

5. *Consider functions f from the whole numbers (non-negative integers) to the whole numbers that have the following properties:*

   - *For all x and y, $f(xy) = f(x)f(y)$,*
   - *$f(30) = 1$, and*
   - *for any n whose last digit is 7, $f(n) = 1$.*

   Obviously, the function whose value at n is 1 for all n is one such function. Are there any others? If not, why not, and if so, what are they?

   Solution: There is one other function satisfying the condition which takes value 0 at 0 and 1 everywhere else. To see that these two are the only examples first note that since $f(2)f(3)f(5) = f(30) = 1$ we must have $f(2) = f(3) = f(5) = 1$. Since $f(2) = f(2 \times 1) = f(2)f(1)$ we also have $f(1) = 1$. If n is a positive integer that is not a multiple of 2 or 5 then for some m, nm has last digit 7. Therefore $f(n)f(m) = f(nm) = 1$ and hence $f(n) = 1$. If $n = 2^a5^bn'$ where n' is not a multiple of 2 or 5 then $f(n) = f(2)^af(5)^bf(n') = 1$. So $f(n)$ must equal 1 for all non-zero n. Finally note that $f(0) = f(0 \times 0) = f(0)f(0)$ so $f(0) = 0$ or $f(0) = 1$. □

6. *$ABCD$ is a quadrilateral having both an inscribed circle (one tangent to all four sides) with center I, and a circumscribed circle with center O. Let S be the point of intersection of the diagonals of $ABCD$. Show that if any two of S, I and O coincide, then $ABCD$ is a square (and hence all three coincide).*

   Solution: We consider the possibilities one at a time.

   If $S = O$ then the two diagonals are diameters of the circumscribed circle, and hence the angles opposite each diagonal are 90°. Thus $ABCD$ is a rectangle, but having an inscribed circle must be a square.

   If $S = I$ then the diagonals are angle bisectors. Then each side subtends the same arc on the circumcircle as its neighbouring sides do, so the four points divide the circumcircle in equal parts, and $ABCD$ is square.

   If $I = O$ then the quadrilateral can be divided into eight right triangles with hypotenuse R (the circumradius) and one leg r (the inradius). These triangles are congruent to one another and the eight angles at O are all equal, hence equal 45°. So these are all isosceles right triangles and again $ABCD$ is a square. □

7. *In a sequence of positive integers an* inversion *is a pair of positions such that the element in the position to the left is greater than the element in the position to the right. For instance the sequence 2,5,3,1,3 has five inversions – between the first and fourth positions, the second and all later positions, and between the third and fourth positions. What is the largest possible number of inversions in a sequence of positive integers whose sum is 2014?*

Solution: We first show that any sequence maximising the number of inversions must be non-increasing. Indeed if there were a consecutive pair a, b in such a sequence with $a < b$, then by exchanging those two elements we do not change the sum, and increase the number of inversions by one. Next we claim that any non-increasing sequence maximising the number of inversions must consist solely of 1's and 2's. Indeed suppose it contained a number $k > 2$. Replace the final k by a pair of elements - $k-1$ in its original position, and 1 in the final position of the sequence. Clearly this does not change the sum. The final 1 is part of an inversion with every element that the original k was except for any other 1's in the sequence. The $k-1$ is likewise part of an inversion with every element that the original k was except any other $k-1$'s in the sequence. So together, they account for at least as many inversions as the original k was part of, together with one more – between each other.

So finally we can assume that the sequence consists of some 2's (say a of them) followed by $n-2a$ 1's (we are interested in the case $n = 2014$). The number of inversions is

$$a(n-2a) = -2a^2 + na = -2(a - n/4)^2 + n^2/8$$

This is maximised when $a - n/4$ is as close to 0 as possible. For $n = 2014$ this means $a = 503$ or $a = 504$. In either case the number of inversions is 503×1008. □

8. *Suppose that a and b are positive integers such that*

$$c = a + \frac{b}{a} - \frac{1}{b}$$

is an integer. Prove that c is a perfect square.

Solution: Simplifying we get $cab = a^2b + b^2 - a$. Since b is a divisor of every term except the last, it must also be a divisor of a. Let $a = bq$ and substitute again to obtain:

$$cb^2q = b^3q^2 + b^2 - bq$$

Now divide by b to give:

$$cbq = b^2q^2 + b - q.$$

Now we can conclude that $b = qt$. Playing the game once more we get:

$$cq^2t = q^4t^2 + qt - q$$

and dividing by q we can conclude that t is a divisor of 1, hence $t = 1$. Unwinding we get $q = b$, $a = bq = b^2$, so $c = b^2 + 1/b - 1/b = b^2$. □

9. *Let ABC be a triangle with $\angle CAB > 45°$ and $\angle CBA > 45°$. Construct an isosceles right angled triangle RAB with AB as its hypotenuse and R inside ABC. Also construct isosceles right angled triangles ACQ and BCP having AC and BC respectively as their hypotenuses and lying entirely outside ABC. Show that $CQRP$ is a parallelogram.*

   Solution: Let $a = |BC|$ be the length of BC, $b = |AC|$ and $c = |AB|$. Then $|AR| = c/\sqrt{2}$, $|AQ| = b/\sqrt{2}$ and $\angle QAR = 45° + (\angle CAB - 45°) = \angle CAB$. So ARQ and ABC are similar triangles and so $|RQ| = a/\sqrt{2}$. Similarly, $|RP| = b/\sqrt{2}$, $|CQ| = b/\sqrt{2}$ and $|CP| = a/\sqrt{2}$. Since opposite sides of $CQRP$ have equal lengths, it is a parallelogram.

   □

10. *Find the largest possible real number C such that for all pairs (x, y) of real numbers with $x \neq y$ and $xy = 2$,*

$$\frac{\left((x+y)^2 - 6\right)\left((x-y)^2 + 8\right)}{(x-y)^2} \geq C.$$

   Also determine for which pairs (x, y) equality holds.

   Solution: Note that $(x+y)^2 = (x-y)^2 + 4xy = (x-y)^2 + 8$. Now let $z = (x-y)^2 > 0$, and we seek to find the minimum value of:

$$\frac{(z+2)(z+8)}{z} = z + 10 + \frac{16}{z} = 10 + 4\left(\frac{z}{4} + \frac{4}{z}\right)$$

   By AM-GM the final factor is at least 2, and so we see that with $C = 18$ the inequality is always fulfilled. Moreover it is equality if $z = 4$, that is $(x-y)^2 = 4$, $(x+y)^2 = (x-y)^2 + 8 = 12$. Considering the four possibilities obtained by taking the positive or negative solution of the two quadratics we see that the relevant solutions are:

$$(\sqrt{3}+1, \sqrt{3}-1),\ (\sqrt{3}-1, \sqrt{3}+1),\ (-\sqrt{3}+1, -\sqrt{3}-1),\ (-\sqrt{3}-1, -\sqrt{3}+1).$$

   □

11. *Show that we cannot find 171 binary sequences (sequences of 0's and 1's), each of length 12 such that any two of them differ in at least four positions.*

   Solution: Let S be any set of binary sequences of length 12 any two of which differ in at least four positions. For each s in S define A_s to be the set of binary sequences that differ from s in at most two positions. If x is any binary sequence of length 12 and it belongs to both A_s and A_t then it must differ from s in exactly two places, likewise from t and these places must be disjoint – otherwise s and t would differ in at most three places. So there can be at most 6 (i.e. 12/2), A_s such that x belongs to A_s. This gives the following inequality:

$$|S|\left(\binom{12}{0} + \binom{12}{1} + \frac{1}{6}\binom{12}{2}\right) \leq 2^{12}.$$

   To see this, allow each binary sequence of length 12 to "vote" for its nearest neighbour in S, if that neighbour is at distance less than 4, and give 1/6 of a vote to each of its nearest neighbours if they are all at distance 4. Then each sequence votes at most once so the vote total is at most 2^{12} but the LHS is the number of elements of S times the maximum number of votes it could receive. So $24|S| \le 2^{12}$, $|S| \le 2^9/3 = 170 + 1/3$. In particular we cannot find 171 such sequences. □

12. *For a positive integer n, let $p(n)$ denote the largest prime divisor of n. Show that there exist infinitely many positive integers m such that $p(m-1) < p(m) < p(m+1)$.*

Solution: We begin with an observation: if q is an odd prime and $a < b$ then the greatest common divisor of $q^{2^a} + 1$ and $q^{2^b} + 1$ is 2. To see this note simply that:

$$q^{2^b} - 1 = (q-1)(q+1)(q^2+1)\cdots(q^{2^{b-1}}+1)$$

so $q^{2^a}+1$ is a divisor of $q^{2^b}-1$. So the g.c.d. must be a divisor of $(q^{2^b}+1) - (q^{2^b}-1) = 2$, hence be 2.

It follows that the integers $q^{2^n}+1$ have infinitely many distinct prime divisors. In particular we can choose k to be the least positive integer such that $p(q^{2^k}+1) > q$. In particular, all prime divisors of $q^{2^j}+1$ are less than q if $j < k$.

Now consider $m = q^{2^k}$. Obviously $p(m) = q$ and by choice $p(m+1) > q$. But the factorization above and the choice made imply that $p(m-1) = p(q^{2^k}-1) < q$.

Since q is an arbitrary odd prime, this gives infinitely many examples. □