New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2013 — Instructions

Solutions due date: 20th October 2013

These problems will be used by the NZMOC to select students for its International Mathematical Olympiad Training Camp, to be held in Auckland between the 12th and 18th of January 2014. Only students who attend this camp are eligible for selection to represent New Zealand at the 2014 International Mathematical Olympiad (IMO), to be held in Cape Town, South Africa in July 2014. The cost of the camp is yet to be determined precisely, but will not be more than $550.

At the camp a squad of 10–12 students will be chosen for further training, and to take part in several international competitions, including the Australian and Asia-Pacific Mathematical Olympiads. The New Zealand team for the 2014 IMO will be chosen from this squad.

Students will be selected into two groups for the camp: juniors and seniors.

- If you are currently in year 12, **or** you have been a member of the NZIMO training squad, then you will be considered only as a senior.
- Otherwise, you will be considered as either a junior or a senior.
- Since the problems and the participants vary from year to year it is hard to be precise about the selection criteria. However, as a rough guide, if you solve five or more of the problems completely then you will be in the running for selection as a junior. Obviously, the criteria for senior selection are somewhat higher.

General instructions:

- Although some problems seem to require only a numerical answer, in order to receive full credit for the problem a complete justification must be provided. In fact, an answer alone will only be worth 1 point out of 7.
- You may not use a calculator, computer or the internet (except as a reference for e.g. definitions) to assist you in solving the problems.
- All solutions must be entirely your own work.
- We do not expect many, if any, perfect submissions. So, please submit all the solutions and partial solutions that you can find.

Students submitting solutions should be intending to remain in school in 2014 and should also hold New Zealand Passports or have New Zealand Resident status. To be eligible for the 2014 IMO you must have been born on or after 10 July 1994, and must not be formally enrolled in a University or similar institution prior to the IMO.

Your solutions, together with a completed Registration Form (overleaf), should be sent to

> Dr Igor Klep, The University of Auckland, Department of Mathematics, Private Bag 92019, Auckland 1142, New Zealand

arriving no later than 20th October 2013. We regret that we are unable to accept electronic submissions. You will be notified whether or not you have been selected for the Camp by mid November 2013.

If you have any questions, please contact Igor Klep (igor.klep@auckland.ac.nz, (09) 373 7599 x84986) or Michael Albert (michael.albert@cs.otago.ac.nz).

September 2013
www.mathsolympiad.org.nz

Registration Form
NZMOC Camp Selection Problems 2013

Name: ______________________________

Gender: male/female **School year level in 2014:** ______________

Home address: ______________________________

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Email address: ______________________________

Home phone number: ______________________________

School: ______________________________

School address: ______________________________

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Principal: ______________ **HOD Mathematics:** ______________

Have you attended the NZMOC January camp before? yes/no

If so, in what year(s)? ______________________________

Have you been a member of the NZMOC training squad? yes/no

If so, in what year(s)? ______________________________

Do you intend to take part in the camp selection problems for any other Olympiad camp? yes/no

If so, and if selected, which camp would you prefer to attend? ______________

Have you put your name forward for a Science camp or any other camp in January? yes/no

Are there any criminal charges, or pending criminal charges against you? yes/no

Some conditions are attached to camp selection. You must be:

- Born on or after 10 July 1994.
- Studying in 2014 at a recognised secondary school in NZ.
- Available in July 2014 to represent NZ overseas as part of the NZIMO team if selected.
- A NZ citizen or hold NZ resident status.

Declaration: I satisfy these requirements, have worked on the questions without assistance from anyone else, and have read, understood and followed the instructions for the January camp selection problems. I agree to being contacted through the email address I have supplied.

Signature: ______________________________ Date: ______________

Attach this registration form to your solutions, and send them to

Dr Igor Klep, The University of Auckland, Department of Mathematics, Private Bag 92019, Auckland 1142, New Zealand,

arriving no later than 20th October 2013.

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1. You have a set of five weights, together with a balance that allows you to compare the weight of two things. The weights are known to be 10, 20, 30, 40 and 50 grams, but are otherwise identical except for their labels. The 10 and 50 gram weights are clearly labelled, but the labels have been erased on the remaining weights. Using the balance exactly once, is it possible to determine what one of the three unlabelled weights is? If so, explain how, and if not, explain why not.

2. Find all primes that can be written both as a sum and as a difference of two primes (note that 1 is *not* a prime).

3. Prove that for any positive integer $n > 2$ we can find n distinct positive integers, the sum of whose reciprocals is equal to 1.

4. Let C be a cube. By connecting the centres of the faces of C with lines we form an octahedron O. By connecting the centers of each face of O with lines we get a smaller cube C'. What is the ratio between the side length of C and the side length of C'?

5. Consider functions f from the whole numbers (non-negative integers) to the whole numbers that have the following properties:

   - For all x and y, $f(xy) = f(x)f(y)$,
   - $f(30) = 1$, and
   - for any n whose last digit is 7, $f(n) = 1$.

   Obviously, the function whose value at n is 1 for all n is one such function. Are there any others? If not, why not, and if so, what are they?

6. $ABCD$ is a quadrilateral having both an inscribed circle (one tangent to all four sides) with center I, and a circumscribed circle with center O. Let S be the point of intersection of the diagonals of $ABCD$. Show that if any two of S, I and O coincide, then $ABCD$ is a square (and hence all three coincide).

7. In a sequence of positive integers an *inversion* is a pair of positions such that the element in the position to the left is greater than the element in the position to the right. For instance the sequence 2,5,3,1,3 has five inversions – between the first and fourth positions, the second and all later positions, and between the third and fourth positions. What is the largest possible number of inversions in a sequence of positive integers whose sum is 2014?

8. Suppose that a and b are positive integers such that

$$c = a + \frac{b}{a} - \frac{1}{b}$$

is an integer. Prove that c is a perfect square.

9. Let ABC be a triangle with $\angle CAB > 45°$ and $\angle CBA > 45°$. Construct an isosceles right angled triangle RAB with AB as its hypotenuse and R inside ABC. Also construct isosceles right angled triangles ACQ and BCP having AC and BC respectively as their hypotenuses and lying entirely outside ABC. Show that $CQRP$ is a parallelogram.

10. Find the largest possible real number C such that for all pairs (x, y) of real numbers with $x \neq y$ and $xy = 2$,

$$\frac{\left((x+y)^2 - 6\right)\left((x-y)^2 + 8\right)}{(x-y)^2} \geq C.$$

Also determine for which pairs (x, y) equality holds.

11. Show that we cannot find 171 binary sequences (sequences of 0's and 1's), each of length 12 such that any two of them differ in at least four positions.

12. For a positive integer n, let $p(n)$ denote the largest prime divisor of n. Show that there exist infinitely many positive integers m such that $p(m-1) < p(m) < p(m+1)$.