



2011 Squad Assignment Four

Algebra

Due: Monday 28th March 2011

1. Find all polynomials $p(x)$ with real coefficients such that

$$p(a + b - 2c) + p(b + c - 2a) + p(c + a - 2b) = 3p(a - b) + 3p(b - c) + 3p(c - a)$$

for all $a, b, c \in \mathbb{R}$.

2. Determine if there exist non-zero real numbers a_1, a_2, \dots, a_{10} such that

$$\left(a_1 + \frac{1}{a_1}\right) \cdot \dots \cdot \left(a_{10} + \frac{1}{a_{10}}\right) = \left(a_1 - \frac{1}{a_1}\right) \cdot \dots \cdot \left(a_{10} - \frac{1}{a_{10}}\right).$$

3. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which satisfy

$$f(x) = \max_{y \in \mathbb{R}} (2xy - f(y))$$

for all $x \in \mathbb{R}$.

Note: In general the expression $a = \max_{s \in S} g(s)$ means that $a \geq g(s)$ for all $s \in S$ and furthermore there exists $s \in S$ such that $a = g(s)$.

4. Given positive real numbers x, y, z , which satisfy $x^2 + y^2 + z^2 + 2xyz = 1$, show that

$$2(x + y + z) \leq 3.$$

5. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function such that $f(f(n)) = 3n$, for all $n \in \mathbb{N}$. Find $f(2011)$.

6. Find all monic polynomials, P , with real coefficients satisfying the following conditions:

(a) $P(x + P(x)) = x^2 + P(P(x))$ for all real x and

(b) $|P(0)| > 1$.

7. Determine all finite sets A of non-negative real numbers, containing at least four distinct elements, and such that for all distinct $a, b, c, d \in A$, $ab + cd \in A$.

8. Let a_1, a_2, \dots, a_n be real numbers with $n \geq 3$, satisfying $a_1 + a_2 + \dots + a_n = 0$, and

$$2a_k \leq a_{k-1} + a_{k+1} \text{ for } k = 2, 3, \dots, n-1.$$

Determine the smallest possible $\lambda(n)$ such that for any such sequence and $k \in \{1, 2, \dots, n\}$, it holds that

$$|a_k| \leq \lambda(n) \cdot \max\{|a_1|, |a_n|\}.$$

February 12, 2011

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