New Zealand Mathematical Olympiad Committee

2011 April Problems — Solutions

These problems are intended to help students prepare for the 2011 camp selection problems, used to choose students to attend our week-long residential training camp in Christchurch in January.

The solutions will be posted in about one month's time, but can be obtained before then by email if you write to one of us with evidence that you've tried the problems seriously.

Good luck!

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1. *Points A,B,C lie on a circle. Line PB is tangent to the circle at B. Let PA_1 and PC_1 be the perpendiculars from P to lines AB and BC respectively (A_1 and C_1 lie on lines AB and BC respectively). Prove that A_1C_1 is perpendicular to AC.*

   Solution: Since $\angle PC_1B = \angle PA_1B = 90°$, the points P,A_1,C_1,B lie on a circle. There are two cases, when they are in order PA_1C_1B or PA_1BC_1. We will consider the order PA_1C_1B and the other case is very similar (the main observation is that they all lie on a circle).

   We have that

   $$\angle CC_1A_1 = 180° - \angle A_1C_1B = \angle A_1PB = 90° - \angle A_1BP.$$

   On the other hand $\angle A_1BP = \angle ACB$. Hence $\angle CC_1A_1 = \angle A_1PB = 90° - \angle ACC_1$, which shows A_1C_1 and AC are perpendicular. The other case is done similarly. □

2. *Does there exist a positive integer k such that all the positive integers from 1 to k can be partitioned into two sets, and all the numbers in each set can be written down one after another (in some order without spaces) to form two new numbers, so that these two numbers are equal.*

   Solution: There does not exist such an integer. Assume such a k exists. It is clear that $k \geq 10$, since the numbers from 1 to 9 never have digits repeating. Consider 10^n, the largest power of 10 before k, i.e. $10^n \leq k < 10^{n+1}$. Then 10^n must be in one of the two sets and hence the constructed number from this set must have n consecutive zeros preceded by a 1.

   Since natural numbers are not written with zeros at the front, this string of n zeros preceded by a 1 could not have formed in the other number by joining two or more other natural numbers, and so it must have come from one natural number. But the next smallest natural number containing such a string is 10^{n+1}, so if the two obtained numbers are equal, we must have used a number greater than or equal to 10^{n+1} in the other set. But this implies $10^{n+1} \leq k$, and so contradicts the maximality of n. □

3. *Nine skiers participated in a race. They started one by one, and each skier completed the race with a constant speed (which could be different for different skiers). Determine if it could happen that each skier participated in an overtaking exactly four times. (In each overtaking, exactly two skiers participated: the one who overtakes and the one who is overtaken.)*

   Solution: This could not happen. Since the speed of each skier is constant, the skier who started first could not overtake anyone. Hence he must have been overtaken four times. Hence he arrived in the fifth position. On the other hand, the skier who started last could not be overtaken by anyone, hence he overtook four other skiers and again arrived in the fifth position, which gives a contradiction. □

4. *Let 100 pairwise distinct real numbers be arranged in a circle. Prove that there exist four consecutive numbers along the circle such that the sum of the middle two numbers is strictly less than the sum of the other two numbers.*

   Solution: *Solution 1.* Assume the contrary. Let the numbers be $a_1, a_2, \ldots, a_{100}$ and use the notation $a_{100+n} = a_n$ (i.e. we work with indices modulo 100). Then for $n = 1, 2, \ldots, 100$ we have the inequality $a_n + a_{n+3} \leq a_{n+1} + a_{n+2}$, which is equivalent to $a_{n+3} - a_{n+2} \leq a_{n+1} - a_n$. This implies $a_{100} - a_{99} \leq a_{98} - a_{97} \leq \ldots \leq a_2 - a_1 \leq a_{100} - a_{99}$. So all of these inequalities are equalities. Hence for $k = 1, 2, \ldots, 50$ we have $a_{2k} - a_{2k-1} = b$ for a fixed b, and similarly for $k = 1, 2, \ldots, 50$ we have $a_{2k+1} - a_{2k} = c$ for a fixed c. Summing these equalities gives $50b + 50c = 0$, so $b = -c$. But then $a_2 - a_1 = b = -c = -(a_3 - a_2) = a_2 - a_3 \Longrightarrow a_1 = a_3$, which contradicts that all numbers are distinct.

   Solution 2. Let a be the smallest number in the circle, b and c its neighbours ($b < c$) and d the other neighbour of b (different from a). Then these numbers go in the order d, b, a, c (or reversed) and $a < d$, $b < c$, so we have $a + b < d + c$, which is what we wanted. □

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