



2011 Squad Assignment Two

Geometry

Due: Monday 28th February 2011

1. Find all possible values of the quotient

$$\frac{r + \rho}{a + b}$$

where r and ρ are respectively the radii of the circumcircle and incircle of the right triangle with legs a and b .

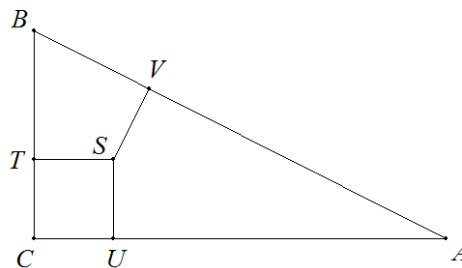
Solution: The distances between the vertices and points of tangency of the incircle (denoted according to the figure below) of a triangle ABC are

$$|AU| = |AV| = \frac{b + c - a}{2}, |BV| = |BT| = \frac{a + c - b}{2}, |CT| = |CU| = \frac{a + b - c}{2}.$$

The points C, T, U and S (the centre of the incircle) form a square. The side of the square is $\rho = |SU| = |CU| = \frac{a+b-c}{2}$. Moreover $r = \frac{c}{2}$ and so

$$r + \rho = \frac{c}{2} + \frac{a + b - c}{2} = \frac{a + b}{2}.$$

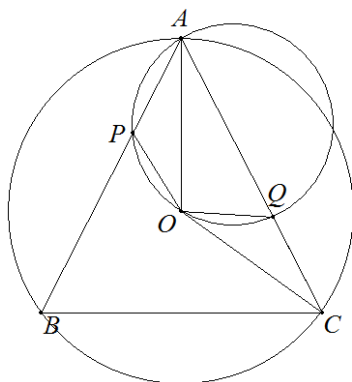
Hence there is only one possible value for $\frac{r+\rho}{a+b}$ which is $\frac{1}{2}$.



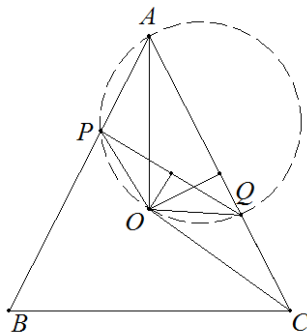
□

2. Let ABC be an isosceles triangle with $|AB| = |AC|$ and P, Q are interior points of AB and AC respectively. Prove that the circumcircle of $\triangle APQ$ passes through the circumcentre of $\triangle ABC$ if and only if $|AP| = |CQ|$

Solution: First assume that the circumcircle of $\triangle APQ$ passes through the circumcentre of the $\triangle ABC$. Let O be the circumcentre of $\triangle ABC$ and join O with P and Q . Because $\triangle ABC$ is isosceles, $\angle PAQ = \angle OAQ$. As $|OA| = |OC|$, we have $\angle OPA = \angle OQC$. This shows that the triangles APO and CQO have the same angles. As $|OA| = |OC|$ this implies that $|AP| = |CQ|$.



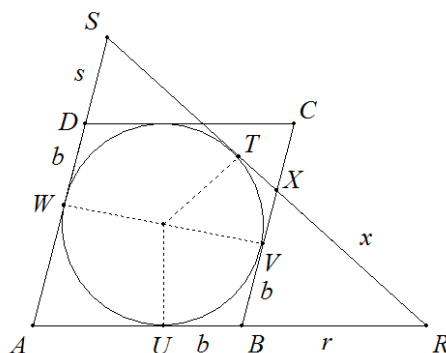
Assume now that $|AP| = |CQ|$ and let O be the intersection of the perpendicular bisectors of PQ and AC (see figure below). First note that the four points A, P, O, Q are concyclic. This is so, because $|OP| = |OQ|$, $|OA| = |OC|$ and $|AP| = |CQ|$ implies that the two triangles APO and CQO are congruent, hence $\angle APO = \angle CQO$, which implies that A, P, O, Q are on a circle.



On the other hand, we have $\angle PAO = \angle QCO$ from the congruence seen above and also $\angle QCO = \angle QAO$ from the definition of O . Therefore, AO is the angle bisector and since $|AB| = |AC|$, the line AO bisects BC at a right angle. This implies that O is the circumcentre of $\triangle ABC$. \square

3. Let $ABCD$ be a rhombus and let a tangent of its incircle cut the interior of the sides BC and CD , and denote R, S the intersections of the tangent with the lines AB, AD respectively. Prove that the value of $|BR| \cdot |DS|$ is independent of the choice of the tangent.

Solution: Let U, V, W, T be the points of tangency of the incircle with the sides AB, BC, DA, DA , and with the tangent respectively. Further let X be the intersection point of the tangent and the side BC , and let $a = |AB| = |AD|, b = |BU| = |BV| = |DW|$ be fixed quantities, while $r = |BR|$ and $s = |DS|$ are the variables dependent on the choice of the tangent. We will show that $r \cdot s = a \cdot b$.



The point R is the centre of similitude of the triangles ARS and BRX . Moreover the incircle is the incircle of $\triangle ARS$ and the excircle of $\triangle BRX$ tangent to BX . According to the well-known fact, the centre of BX is the centre of symmetry of the points of tangency of the incircle and excircle with BX . This means that the ratio $|SW| : |AR|$ in $\triangle ARS$ corresponds to the ratio $|BV| : |BR|$ in $\triangle BRX$, that is

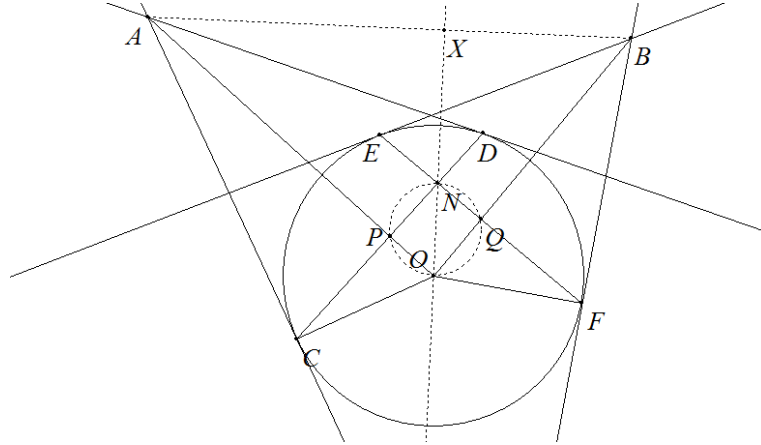
$$\frac{b+s}{a+r} = \frac{b}{r},$$

which is equivalent to $r \cdot s = a \cdot b$. □

4. In an acute-angled triangle ABC , M is the midpoint of side BC , and D, E and F the feet of the altitudes from A, B and C , respectively. Let H be the orthocentre of triangle ABC , S the midpoint of AH , and G the intersection of FE and HA . If N is the intersection of the line segment AM and the circumcircle of triangle BCH , prove that $\angle HMA = \angle GNS$.

Solution: *First solution.* Let the line AN meet the circle BHS again at the point A' . The $ABA'C$ is a parallelogram and $\angle HCA' = \angle HCB + \angle BCA' = \angle HCB + \angle ABC = 90^\circ$, so the points A, B, N all lie on the circle with diameter HA' and therefore $\angle ANH = 90^\circ$.

hence $OA \cdot OP = OC^2$. Similarly we obtain $OB \cdot OQ = OF^2$ and since $OC = OF$, we have $OA \cdot OP = OB \cdot OQ$. It follows therefore that the triangles OAB and OQP are similar, so $\angle OQP = \angle OAB$.

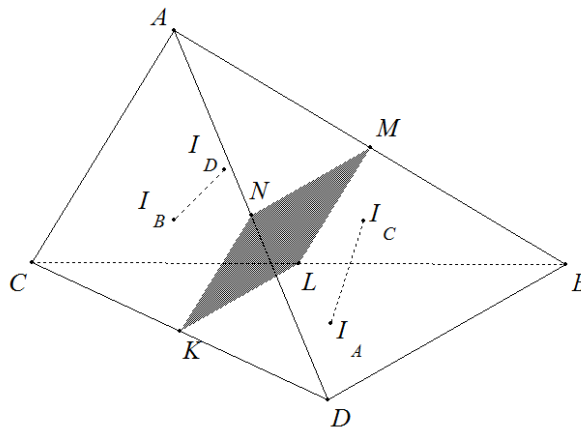


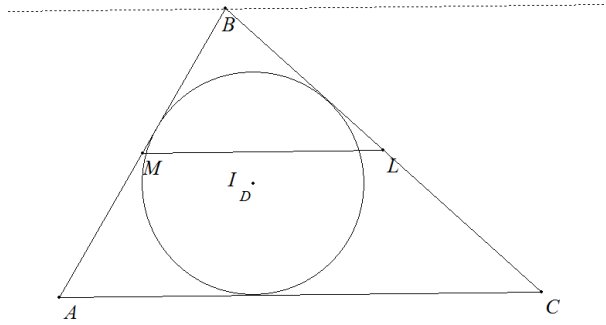
On the other hand, since $\angle OPN = \angle OQN = 90^\circ$, we infer that the quadrilateral $OPNQ$ is cyclic. Hence we obtain $\angle ONP = \angle OQP = \angle OAB$. Let X be the point of intersection of the lines ON and AB . Then $\angle OAB = \angle ONP = 180^\circ - \angle PNX$, which implies that the quadrilateral $APNX$ is cyclic. Therefore, since $\angle APN = 90^\circ$, we immediately get $\angle AXN = 90^\circ$. \square

6. Can the four incentres of the four faces of a tetrahedron be coplanar?

Solution: They cannot. Let I_A, I_B, I_C, I_D be the incentres of the triangles BCD, ACD, ABD, ABC respectively. Assume they are coplanar. Then in their common plane they form either a convex quadrilateral, or a triangle inside of which the remaining point lies.

Case 1. Without loss of generality assume that I_A, I_B, I_C, I_D is a convex quadrilateral. Then the line segments $I_A I_C$ and $I_B I_D$ intersect. Let M, N, K, L be the midpoints AB, AD, CD, BC respectively. We observe that M, N, K, L are coplanar, since MN and KL are both parallel to BD , and this plane separates the tetrahedron with A, C on one side and B, D on the other.





Consider triangle ABC and let ℓ be the line through B parallel to AC . Then the incircle of triangle ABC lies between ℓ and AC and is tangent to AC but not to ℓ . This implies that the centre of the incircle of triangle ABC , I_D , and B lie on different sides of the line ML , which is exactly half way between AC and ℓ . By a similar process we see that I_B and I_D lie on the same side of the plane $MNKL$ (on the opposite side of points B, D) while I_A and I_C lie on the other side. Hence the line segments $I_A I_C$ and $I_B I_D$ cannot intersect and we have reached a contradiction.

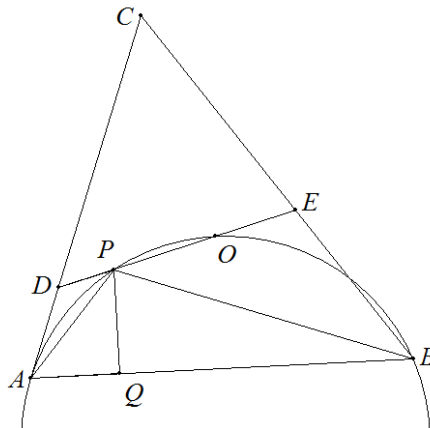
Case 2. It is sufficient to show that I_A cannot lie inside the triangle $I_B I_C I_D$. This follows from the fact that the points I_B, I_C, I_D lie strictly to one side of the plane BCD while I_A lies inside this plane. \square

7. Let O be the circumcentre of an acute-angled triangle ABC . A line through O intersects the sides CA and CB at points D and E respectively, and meets the circumcircle of triangle ABO again at point $P \neq O$ inside the triangle. A point Q on side AB is such that

$$\frac{AQ}{QB} = \frac{DP}{PE}.$$

Prove that $\angle APQ = 2\angle CAP$.

Solution: Denote $\angle PAD = \phi, \angle QPA = \psi, \angle BCA = \gamma$. The $\angle APB = 2\gamma$ and $\angle DAP + \angle EBP = \angle APB - \angle ACB = \gamma$, so $\angle PBE = \gamma - \phi$ and $\angle BPQ = 2\gamma - \psi$. Since $\angle APD = \angle BPE = 90^\circ - \gamma$, we also have $\angle ADP = 90^\circ + \gamma - \phi$ and $\angle BEP = 90^\circ + \phi$.



The sine rule in triangles APD and PBE gives us

$$\frac{DP}{QB} = \frac{DP}{PA} \cdot \frac{PA}{PB} \cdot \frac{PB}{PE} = \frac{\sin \phi \cos \phi}{\sin(\gamma - \phi) \cos(\gamma - \phi)} \cdot \frac{PA}{PB} = \frac{\sin 2\phi}{\sin(2\gamma - 2\phi)} \cdot \frac{PA}{PB}.$$

On the other hand,

$$\frac{AQ}{QB} = \frac{AQ}{AP} \cdot \frac{AP}{BP} \cdot \frac{BP}{QB} = \frac{\sin \psi}{\sin(2\gamma - \psi)} \cdot \frac{AP}{PB}.$$

Now $\frac{AQ}{QB} = \frac{DP}{PE}$ implies $f(2\phi) = f(\psi)$, where $f(x) = \frac{\sin x}{\sin(2\gamma - x)}$. However f is strictly increasing for all x because $f'(x) = \frac{\sin(2\gamma)}{(\sin(2\gamma - x))^2} > 0$, so it follows that $\psi = 2\phi$. \square

March 9, 2011

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