New Zealand Mathematical Olympiad Committee

2011 Squad Assignment One

Combinatorics

Due: Monday 14th February 2011

1. *A tennis tournament has at least three participants. Every participant plays exactly one match against every other participant, and moreover every participant wins at least one of his or her matches. (Draws do not occur in tennis.)*

   Show that there are three participants A, B, C for which the following holds: A wins against B, B wins against C, and C wins against A.

   Solution: We will present four different solutions. They are all essentially proofs by induction, although only two are explicitly stated in this form: the other two make use of the "extremal principle".

   Solution 1. We first show that there is a cycle, i.e. m distinct participants $A_1, \ldots, A_m$ such that A_1 beat A_2, A_2 beat A_3, and so on, up to A_m, who beat A_1. Choose an arbitrary participant B_1. Then B_1 beat some participant B_2, who beat some B_3, and so on. Since there are only finitely many participants this sequence must eventually contain a player B_t who beat some B_u, with $u < t-1$. Then $B_u, B_{u+1}, \ldots, B_t$ form a cycle, which necessarily has length at least 3.

   We now show that there must in fact be a cycle of length 3. Among all cycles chose $C_1, C_2, \ldots, C_M$ of minimal length M. If $M = 3$ then we are done; otherwise, $M > 3$ and we consider the game between C_1 and C_3. If C_3 beat C_1 then C_1, C_2, C_3 is a cycle of length 3, contradicting the minimality of M. On the other hand, if C_1 beat C_3, then by omitting C_2 we obtain a cycle $C_1, C_3, \ldots, C_M$ of length $M-1$, again contradicting the minimality of M. Thus, we must in fact have $M = 3$, and C_1, C_2, C_3 is the required cycle of length 3.

   Solution 2. Choose a player A who won the least number of games, and consider the set L of players that lost to A. The set L is nonempty, since each player won at least one game, so we may choose a player B belonging to L.

   If B only won against players belonging to L then B won at most $|L| - 1 < |L|$ games, contradicting our choice of A as a player who won the least games. So B must have beaten some player C not belonging to L. Then C must have beaten A, so A, B, C form the required three-cycle.

   Solution 3. We will prove the result by strong induction. For the base case $n = 3$, choose an arbitrary player A. Then A must beat some player B, and B in turn must beat the third player C. Then finally C must beat A, since C wins at least one game and lost to B. This gives us a three-cycle.

   Now suppose that the result is true for any tournament satisfying the given conditions with $3 \leq k \leq n$ players, and consider such a tournament with $n + 1$ players. Choose an arbitrary player A, and let W be the set of players that won against A, and L the set of players that lost against A. As in Solution 2 above L must be nonempty.

If there is a player B in L that beat a player C in W then A, B, C form the required three-cycle, and we are done. Suppose then that there is no such player, and consider the tournament consisting of just the players in L. Since each player in L beat at least one other player, and did not beat either A or a player in W, each player in L must have beaten some other player in L. Moreover, for this to be possible there must be at least 3 players in L. The induction hypothesis then gives us a three-cycle within L, and we are done.

Solution 4. Suppose there is no three-cycle, We will show that it is possible to label the players P_1, P_2, ..., P_n in such a way that P_i beat P_j if $i < j$. Then P_n will have lost all of his or her games, contradicting the fact that every player wins at least one game.

To construct the labelling, choose an arbitrary player as P_1. Suppose now that for some $1 \le k \le n-1$ we have chosen k players $P_1, \dots, P_k$ such that P_i beat P_j if $i < j$. Each player won at least one game, and since $P_1, \dots, P_{k-1}$ all beat P_k, there must be some as-yet unlabelled player Q such that P_k won against Q. If $i < k$ then P_i must also have won against Q, otherwise P_1, P_k, Q form a three-cycle, and this implies that we may choose Q as player P_{k+1}. By induction this gives us the required labelling, and the problem is solved. □

2. *There are n towns, some of which are connected by a total of m two-way air routes. For $i = 1, 2, \dots, n$, let d_i be the number of routes going from town i. If $1 \le d_i \le 2010$ for each $i = 1, 2, \dots, n$, prove that*

$$\sum_{i=1}^{n} {d_i}^2 \le 4022m - 2010n.$$

Find all n for which equality can be attained.

Solution: By the given conditions $0 \le (d_i - 1)(2010 - d_i)$ holds for each i, so ${d_i}^2 \le 2011d_i - 2010$. Since $\sum_{i=1}^{n} d_i = 2m$, summing up these inequalities gives

$$\sum_{i=1}^{n} {d_i}^2 \le 2011 \cdot \sum_{i=1}^{n} d_i - 2010n = 4022m - 2010n,$$

as desired.

Equality holds if and only if $d_i \in \{1, 2010\}$ for every $1 \le i \le n$. This splits into two cases:

If $n = 2k$ for some $k \in \mathbb{N}$, then setting an airline between towns i and $i+k$ for $i = 1, \dots, k$ and no other airlines yields a configuration with $d_i = 1$ for all i.

If $n = 2k - 1$ for some $k \in \mathbb{N}$, we cannot have $d_i = 1$ for all i because the sum of all d_i must be even, so we must have $d_j = 2010$ for some j; hence $n \ge 2011$. On the other hand, setting an airline between towns $2i$ and $2i+1$ for $i = 1006, \dots, k-1$ and between 1 and i for $1 \le i \le 2011$ yields a configuration with $d_1 = 2010$ and $d_i = 1$ for $i = 2, \dots, n$.

Therefore equality can be attained if and only if n is even or $n \ge 2011$. □

3. *The cells of an $n \times n$ table are to be filled with the numbers 1, 2, 3 and 4 in such a way that whenever four cells share a common vertex they are to contain all four numbers. How many ways are there to fill in the table?*

Solution: We first show that the table is correctly filled in if and only if at least one of the following conditions is fulfilled:

(a) Each row of the table has exactly two numbers alternating throughout the entire row. One pair of numbers appears in the even numbered rows, and the other pair in the odd numbered rows.

(b) Each column of the table has exactly two numbers alternating throughout the entire column. One pair of numbers appears in the even numbered columns, and the other pair in the odd numbered columns.

To prove this, we may assume without loss of generality that the upper left 2×2 square is filled in as follows:

1	2	$\cdots$
3	4	$\cdots$
$\vdots$	$\vdots$	$\ddots$

Consider the first two columns. The two entries in the third row must be 1 and 2 in some order, and then in turn the entries in the fourth row must be 3 and 4 in some order. Continuing in this way we see inductively that the two entries in the odd rows must be 1 and 2, in either order, and the entries in the even rows must be 3 and 4, in either order. Moreover, any way of filling in the first two columns satisfying this rule satisfies the condition given in the problem.

Suppose now that the first $k \geq 2$ columns have been filled in satisfying the given rule. We make several observations:

(a) As soon as one cell of the $(k+1)$th column is specified there is at most one way to complete the rest of the column. This is because the neighbours of any filled in cell will now be adjacent to at least three filled in cells, leaving at most one possible number that may be entered.

(b) There is always at least one way to fill in the $(k+1)$th column, namely, to simply copy the $(k-1)$th.

(c) If the kth column alternates then there are exactly two ways to complete the kth column; otherwise, there is just one.

To prove this last observation, first suppose without loss of generality that the kth column alternates 1 and 2. Then each square in the $(k+1)$th column may only contain 3 or 4, so there are at most two ways to complete the column, by the first observation. Moreover, alternating 3 and 4, starting with either one, clearly satisfies the condition, giving us exactly two ways to complete it.

Next suppose that the kth column does not alternate. Then it must contain three consecutive entries that are all different (for example, the first entry that does not equal either of the entries in the first two rows, and the two preceding entries).

Without loss of generality this gives us the configuration

1	
2	x
3	

,

and there is at most one way to complete the $(k+1)$th column, because x must be 4. By the second observation the $(k+1)$th column is therefore a copy of the $(k-1)$th.

Putting these observations together, if the first of our two columns filled in as above doesn't alternate 1 and 3, then the second column won't alternate 2 and 4, and each column will be equal to the column two before. This implies that the rows alternate. On the other hand, if the first column does alternate, then the next will too, and after that there are two possible ways to fill in each column, each of which alternates. This gives the claim made above.

The number of ways to complete the table are now easily counted. There are 4! ways to complete the top left 2×2 square, and for each of these there are 2^{n-2} ways to complete the table so that the rows alternate (the first entry of rows 3–n may be chosen in two possible ways, and then the table is determined), and similarly 2^{n-2} ways to complete the table so that the columns alternate. Since there is just one way to complete the table so that both rows and columns alternate, this gives a total of

$$4!(2 \times 2^{n-2} - 1) = 4!(2^{n-1} - 1)$$

ways to complete the table. □

4. *There are $2n$ people seated around a circular table, and m cookies are distributed among them. The cookies may be passed around under the following rules:*

   - *Each person may only pass cookies to his or her neighbours.*
   - *Each time someone passes a cookie, he or she must also eat a cookie.*

   Let A be one of these people. Find the least m such that no matter how m cookies are distributed to begin with, there is a strategy to pass cookies so that A receives at least one cookie.

   Solution: We claim that the minimum possible value is $m = 2^n$. To assist in solving this problem we will define a monovariant — a quantity that can either only increase, or only decrease, as the cookies are passed around. The use of the monovariant to prove $m \geq 2^n$ is necessary follows the official solution, and the use of it to prove $m \geq 2^n$ is sufficient follows M. Granville.

   We begin by labelling the people A_{-n+1}, A_{-n+2}, …, A_{-1}, A_0, A_1, …, A_{n-1}, A_n in a clockwise fashion, in such a way that $A = A_0$. For convenience we also let $A_{-n} = A_n$. We now assign weight $1/2^{|i|}$ to a cookie held by person A_i, and letting a_i be the number of cookies held by A_i we define

   $$W = \sum_{i=-n+1}^{n} \frac{a_i}{2^{|i|}}$$

   to be the total weight of the cookies. We note that if a cookie is passed *towards* A_0 (from $A_{\pm i}$ to $A_{\pm(i-1)}$, for i positive) then the total weight W is unchanged, because two cookies of weight $1/2^i$ become a single cookie of weight $1/2^{i-1}$. On the other hand, if a cookie is passed *away* from A_0, then the total weight decreases. So W is non-increasing.

Suppose first that $m < 2^n$, and that all of the cookies are initially given to A_n. Then $W = m/2^n < 1$ initially, and it is impossible for $A = A_0$ to receive a cookie, because then the final weight would be at least 1.

We must now show that if $m \geq 2^n$, then no matter how the cookies are distributed there is a strategy for passing them such that A receives a cookie. We will show that this can always be done just by passing the cookies around one side of the circle, and to do this we modify W, letting

$$W_+ = \sum_{i=0}^{n} \frac{a_i}{2^i}, \qquad\qquad W_- = \sum_{i=0}^{n} \frac{a_{-i}}{2^i}.$$

Then W_+ is the total weight of the cookies held by $A_0, A_1, \ldots, A_n$, and W_- is the total weight of the cookies held by $A_0, A_{-1}, \ldots, A_{-n}$. We now claim:

Lemma 4.1. *If $m \geq 2^n$ then W_+, W_- can't both be smaller than 1.*

Proof. We have

$$\begin{aligned} W_+ + W_- &= 2a_0 + 2\frac{a_n}{2^n} + \sum_{i=1}^{n-1} \frac{a_i + a_{-i}}{2^i} \\ &\geq \frac{a_0}{2^{n-1}} + \frac{a_n}{2^{n-1}} + \sum_{i=1}^{n-1} \frac{a_i + a_{-i}}{2^{n-1}} \\ &= \frac{1}{2^{n-1}} \sum_{i=-n+1}^{n} a_i \\ &= \frac{m}{2^{n-1}} \geq 2. \end{aligned}$$

The lemma follows. □

We may assume without loss of generality that $W_+ \geq 1$. We now ask $A_n, A_{n-1}, \ldots, A_1$ in turn to pass as many cookies as they can to their anticlockwise neighbour: first A_n passes $\lfloor a_n/2 \rfloor$ cookies to A_{n-1}, then, once A_i has been passed cookies by A_{i+1}, he or she passes as many cookies as possible to A_{i-1}. Let x_i be the number of cookies held by A_i once this is complete. Then $x_i \in \{0, 1\}$ for $i = 1, \ldots, n$, and the total weight W_+ will be unchanged, because cookies have only been passed towards A_0.

To complete the proof we show that $x_0 \geq 1$. We have

$$
\begin{aligned}
1 \leq W_+ &= \sum_{i=0}^{n} \frac{x_i}{2^i} \\
&= x_0 + \sum_{i=1}^{n} \frac{x_i}{2^i} \\
&\leq x_0 + \sum_{i=1}^{n} \frac{1}{2^i} && \text{(since } x_i \in \{0,1\} \text{ for } i = 1, \ldots, n) \\
&< x_0 + \sum_{i=1}^{\infty} \frac{1}{2^i} = x_0 + 1.
\end{aligned}
$$

So $x_0 > 0$, and since x_0 is an integer, we are done. □

5. *A group of students at a school is* popular *if any other student at the school has a friend in the group. Suppose it is known that the school has at least 100 popular groups. Show that it must in fact have at least 101 popular groups.*

   You may assume that friendship is symmetric: if A is friends with B, then B is friends with A.

   Solution: It suffices to show that the number of popular groups must be odd. Let S be the set of all students at the school, and let V be the set of non-empty subsets of S. We will show that the number of popular groups is odd by constructing a graph with vertex set V in which the vertices of even degree are precisely the popular groups. The result will then follow from the handshake lemma.

   To construct the graph, we will join A and B in V by an edge if $A \cap B = \emptyset$ and no-one in A is friends with anyone in B. We now consider the degree of each vertex A. If A is popular and B is disjoint from A then every student in B is friends with a student in A, so there is no edge between A and B. It follows that all popular groups have degree 0. Conversely, if A has degree 0 and there is a student s such that $s \notin A$, then the fact that there is no edge between A and $\{s\}$ implies that s is friends with someone in A, so A is popular.

   Now suppose that A is not popular, and let C be the (nonempty) set of students in $S \setminus A$ that have no friends in A. Then for any $B \in V$, there is an edge between A and B if and only if $B \subseteq C$; since $\emptyset \notin V$ the number of such sets is $2^{|C|} - 1$, which is odd. Thus, all non-popular groups have odd degree, and all popular groups have even degree, as claimed.

   By the handshake lemma there is an even number of vertices of odd degree, and so the number of non-popular groups is even. But since $|V| = 2^{|S|} - 1$ is even, this means that the number of popular groups is odd. □

6. *A certain country has n cities, some of which are connected by one-way roads. Any given pair of cities can have more than one road between them, in either or both directions.*

   It is known that any two routes from the capital Alphaton to the largest city Omegaville via these roads must have at least one road in common. Show that some road must belong to all of the routes from Alphaton to Omegaville.

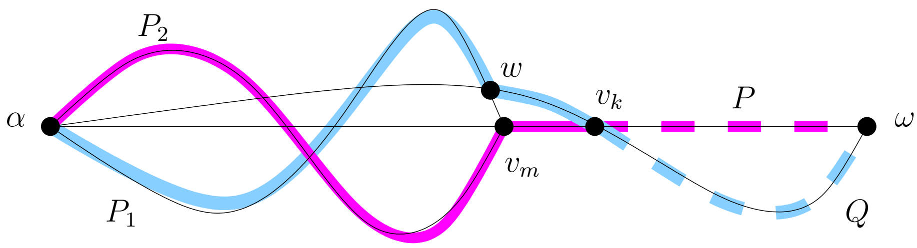

Figure 1: Construction of the paths R_1, R_1' (pale blue) and R_2, R_2' (magenta). The dashed paths (R_1', R_2') omit the dashed portions.

Solution: The cities and roads form a "directed graph" or *digraph*, with the cities as vertices, and a directed edge from u to v if there is a one way road from city u to city v. Label the vertices representing Alphaton and Omegaville α and ω respectively (natually enough!). Our strategy will be to identify a candidate for an edge that belongs to all possible routes, and then show that it really does do what we want. The tool we will use for doing this is the extremal principle, and the underlying idea is this:

> Find two paths that go as far as possible without sharing an edge. Then the next edge on these paths must belong to all routes, or we could find two paths that go further without sharing an edge.

In order to apply this idea we first need a way to measure the distance travelled from α towards ω. The yardstick we will use is the shortest path from α to ω, so this is the first thing we'll define.

Let P be a path of minimal length from α to ω, with vertices $\alpha = v_0, v_1, \ldots, v_n = \omega$, and edges e_i from v_{i-1} to v_i. The fact that P is of minimal length guarantees that the vertices v_i are distinct: if we had $v_i = v_j$ for some $i < j$ then we could obtain a shorter path from α to ω by following P from v_0 to $v_i = v_j$ and then following P from v_j to v_n. Let m be the largest integer such that there are edge disjoint paths from v_0 to v_m, and observe that we must have $m < n$, by the statement of the problem. We claim that the edge $e = e_{m+1}$ from v_m to v_{m+1} lies on all paths from α to ω.

Suppose to the contrary that there is a path Q from α to ω that does not use the edge e. Let P_1 and P_2 be two edge disjoint paths from v_0 to v_m, and let w be the last vertex on Q that belongs to either P_1 or P_2 (note that w must exist, since α lies on all three paths). Without loss of generality we may assume P_1 and P_2 have no repeated vertices (by the argument we used for P), and that w lies on P_1. We construct two paths from α to ω as follows: we let R_1 be the path that follows P_1 as far as w, and then follows Q from w to ω; and we let R_2 be the path that follows P_2 from α to v_m, and then follows P from v_m to ω. See Figure 1.

Now, let v_k be the first vertex among $v_{m+1}, \ldots, v_n = \omega$ encountered on R_1, and consider the paths R_1', R_2' obtained by following R_1 and R_2 until v_k is first encountered. We claim that R_1' and R_2' are edge disjoint, contradicting the choice of v_m as the furthest vertex from α on P that can be reached by two edge disjoint paths.

To prove the claim we consider the different portions of R_1' and R_2'. Firstly, R_1' cannot share any edges with the portion of R_2' from v_{m+1} to v_k, because v_k is the only vertex on this path belonging to R_1'. Additionally, R_1' doesn't use e_{m+1}, because W doesn't use this edge, and if P_1 did, then it would have v_m as a repeated vertex. Finally, R_1' cannot share an edge with the portion of R_2' coming from P_2, because P_1 is edge disjoint from P_2, and the portion of Q from w to v_k has no vertices in common with P_2 except possibly w. This proves the claim.

The existence of Q contradicts our choice of m, and so there can be no path from α to ω that omits e_{m+1}. This completes the proof. $\square$

7. *Let $X = \{A_1, A_2, \dots, A_n\}$ be a set of distinct 3-element subsets of $\{1, 2, \dots, 36\}$ such that*

   (a) A_i and A_j have non-empty intersection for every i, j.

   (b) The intersection of all the elements of X is the empty set.

   Show that $n \leq 100$. How many such sets X are there when $n = 100$?

   Solution: For a problem like this, a useful first step can be to try to identify the configurations that realise the bound of $n = 100$. This can give you an idea of what you're aiming for, and how to prove it. In addition, if you don't manage to prove the bound, you may still pick up any points available for correctly counting the configurations that realise it.

   After playing around for a bit you may find one or both of the following two ways of realising the bound:

   (I) The sets $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, together with all subsets of the form $\{1, x, y\}$ with $x \in \{2,3,4\}$ and $y \in \{5, \dots, 36\}$; a total of $4 + 3 \times 32 = 100$ sets.

   (II) The set $\{1,2,3\}$, together with all subsets of the form $\{x,2,3\}$, $\{1,x,3\}$ and $\{1,2,x\}$ for $x \in \{4, \dots, 36\}$; a total of $1 + 3 \times 33 = 100$ sets.

   Even if you only find one of these configurations, this can still give you a way to attack the problem, and trying to prove that the configuration you've found is the only possibilty should lead you to the second. This is in fact what happened to me when I first solved this problem: initially I only found the second type of configuration given above, and discovered the first while trying to prove that it was impossible for X to be large if it contained sets A_i, A_j, A_k such that $A_i \cap A_j = A_j \cap A_k = A_k \cap A_i = \{l\}$.

   A feature that the two configurations above share is the existence of a *triangle:* sets A_i, A_j, A_k such that

   $$A_i \cap A_j = \{x\}, \qquad A_j \cap A_k = \{y\}, \qquad A_k \cap A_i = \{z\}$$

   for three distinct elements $x, y, z \in \{1, 2, \dots, 36\}$, which we will call the *vertices* of the triangle. Moreover, the triangle gives us a way to home in on the "special" elements of the configuration: in the first one of the "sides" must be $\{2,3,4\}$, with 1 as the remaining vertex, and in the second configuration any triangle will have vertices 1, 2 and 3. Finding a way to identify these points from the start will surely help in determining the configurations that realise the bound, so let's try to show that for X to be large it must contain a triangle, and then try to exploit the triangle to prove the bound.

   To show that X must have a triangle, we prove:

Lemma 7.1. *If X does not contain a triangle then $|X| < 100$.*

Proof. First note that if X is triangle-free then it cannot contain sets A_i, A_j, A_k such that $A_i \cap A_j = A_j \cap A_k = A_k \cap A_i = \{x\}$. This is because X must also contain a set A_m that omits x but meets each of A_i, A_j, A_k, and then A_i, A_j and A_m will form a triangle, because the elements in common between A_m and each of A_i, A_j, A_k will all be distinct.

Suppose next that X contains sets A_i, A_j such that $|A_i \cap A_j| = 1$. Without loss of generality we may assume that these sets are $A_1 = \{1, 2, 3\}$ and $A_2 = \{1, 4, 5\}$. Now X must contain some set A_k that does not contain 1, and any such set must meet both A_1 and A_2, so must contain at least one of 2 and 3, and at least one of 4 and 5. In addition, it must contain either $\{2, 3\}$ or $\{4, 5\}$, or A_1, A_2 and A_k will form a triangle. This means that X contains at most four sets that omit 1, and we may assume that one of them is $A_3 = \{2, 3, 4\}$.

Applying the same argument to A_2 and A_3 we see that there are at most four sets that omit 4. Any further sets belonging to X must then contain $\{1, 4\}$; there are at most 34 such sets, so X certainly can't contain more than

$$2 + 4 + 4 + 34 = 44 < 100$$

sets.

Lastly we must consider the case where no two sets in X intersect in a set of size 1. Without loss of generality $\{1, 2, 3\} \in X$. Then X must contain some set omitting 1 but meeting $\{1, 2, 3\}$ in a set of size 2; without loss of generality this set is $\{2, 3, 4\}$. Additionally we require a set omitting 2, and this must meet $\{1, 2, 3\}$ in $\{1, 3\}$, and $\{2, 3, 4\}$ in $\{3, 4\}$, so can only be $\{1, 3, 4\}$. A similar argument forces $\{1, 2, 4\}$ to belong to X, and at this point we can add no further sets without violating the condition $|A_i \cap A_j| > 1$. In this case X contains at most four sets (which can be regarded as the faces of a tetrahedron), so the lemma is proved. □

We now exploit the triangle.

Lemma 7.2. *Suppose that X contains a triangle with vertices a, b, c, and let X_a be the set of elements of X that contain a but not b or c, and X_{bc} the set of elements of X that contain b and c but not a. Then*

$$|X_a| + |X_{bc}| \leq 33,$$

with equality if and only if one of the following conditions holds:

(a) There is $d \notin \{a, b, c\}$ such that $X_{bc} = \big\{\{b, c, d\}\big\}$ and $X_a = \big\{\{a, d, x\} : x \notin \{a, b, c, d\}\big\}$;

(b) $X_a = \emptyset$ and $X_{bc} = \big\{\{b, c, x\} : x \notin \{a, b, c\}\big\}$.

In fact this lemma doesn't require a, b, c to be the vertices of a triangle, just that $|X_{bc}| > 0$. Note further that at this stage we do not assume that $\{a, b, c\} \in X$.

Proof. Since X contains a triangle with vertices a, b, c there must be some $d \notin \{a, b, c\}$ such that $\{b, c, d\} \in X_{bc}$. Elements of X_a must therefore have the form $\{a, d, x\}$, for $x \notin \{a, b, c, d\}$. We consider two cases, according to whether or not $\{b, c, d\}$ is the only element of X_{bc}.

If $\{b, c, d\}$ is the only element of X_{bc} then every set of the form $\{a, d, x\}$ may belong to X_a. This gives us at most 32 sets in X_a, with the equality $|X_a| + |X_{bc}| = 33$ realised if and only if X_a contains all such sets. This is condition (a) above.

If X_{bc} contains a second set $\{b, c, e\}$ and no other then X_a may contain only the set $\{a, d, e\}$, and $|X_a| + |X_{bc}| \leq 3$. If X_{bc} contains any additional sets then X_a must be empty, and in this case $|X_a| + |X_{bc}|$ is maximised when X_{bc} contains all sets of the form $\{b, c, x\}$, as in condition (b) above. ☐

Suppose now that X contains a triangle, and assume without loss of generality that the sides of the triangle are $\{2, 3, 4\}$, $\{1, 3, 5\}$, and $\{1, 2, 6\}$. If $A_i \in X$ does not belong to one of the six sets $X_1, X_2, X_3, X_{12}, X_{13}, X_{23}$ then A_i must equal either $\{1, 2, 3\}$ or $\{4, 5, 6\}$. We cannot have both of these sets in X, since they are disjoint, so

$$|X| \leq 1 + |X_1| + |X_2| + |X_3| + |X_{12}| + |X_{13}| + |X_{23}| \leq 100.$$

This establishes the bound. For equality to hold we must have

$$|X_1| + |X_{23}| = |X_2| + |X_{13}| = |X_3| + |X_{12}| = 33,$$

and so either condition (a) or (b) of Lemma 7.2 must hold for each. However, condition (a) can hold for at most one vertex of the triangle: if it holds for both 1 and 2, then the disjoint sets $\{1, 4, 7\}$ and $\{2, 5, 8\}$ will belong to X, contradicting the statement of the problem.

Suppose first that conditon (a) holds with say $a = 1$. Then condition (b) must hold for the vertices 2 and 3. Moreover X can contain $\{1, 2, 3\}$ but not $\{4, 5, 6\}$, since $\{4, 5, 6\}$ is disjoint from $\{1, 2, 7\}$, which must belong to X. This leads to the family of sets (I) above. On the other hand, if condition (a) does not hold for any vertex then condition (b) must hold for each. Again X may contain $\{1, 2, 3\}$ but not $\{4, 5, 6\}$, and we arrive at the family of sets (II).

It remains to count the number of ways to realise the bound. Families of type (I) are completely determined by the choice of vertex 1 and opposite side $\{2, 3, 4\}$, so there are $36\binom{35}{3}$ such families, while families of type (II) are completely determined by the choice of vertices $\{1, 2, 3\}$. This gives a total of

$$36\binom{35}{3} + \binom{36}{3} = 34\binom{36}{3}$$

families altogether. ☐

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