# New Zealand Mathematical Olympiad Committee

## 2011 Squad Assignment One

*Combinatorics*

### Due: Monday 14th February 2011

1. A tennis tournament has at least three participants. Every participant plays exactly one match against every other participant, and moreover every participant wins at least one of his or her matches. (Draws do not occur in tennis.)

   Show that there are three participants $A$, $B$, $C$ for which the following holds: $A$ wins against $B$, $B$ wins against $C$, and $C$ wins against $A$.

2. There are $n$ towns, some of which are connected by a total of $m$ two-way air routes. For $i = 1, 2, \ldots, n$, let $d_i$ be the number of routes going from town $i$. If $1 \leq d_i \leq 2010$ for each $i = 1, 2, \ldots, n$, prove that

   $$\sum_{i=1}^{n} {d_i}^2 \leq 4022m - 2010n.$$

   Find all $n$ for which equality can be attained.

3. The cells of an $n \times n$ table are to be filled with the numbers 1, 2, 3 and 4 in such a way that whenever four cells share a common vertex they are to contain all four numbers. How many ways are there to fill in the table?

4. There are $2n$ people seated around a circular table, and $m$ cookies are distributed among them. The cookies may be passed around under the following rules:

   - Each person may only pass cookies to his or her neighbours.
   - Each time someone passes a cookie, he or she must also eat a cookie.

   Let $A$ be one of these people. Find the least $m$ such that no matter how $m$ cookies are distributed to begin with, there is a strategy to pass cookies so that $A$ receives at least one cookie.

5. A group of students at a school is *popular* if any other student at the school has a friend in the group. Suppose it is known that the school has at least 100 popular groups. Show that it must in fact have at least 101 popular groups.

   You may assume that friendship is symmetric: if $A$ is friends with $B$, then $B$ is friends with $A$.

6. A certain country has $n$ cities, some of which are connected by one-way roads. Any given pair of cities can have more than one road between them, in either or both directions.

   It is known that any two routes from the capital Alphaton to the largest city Omegaville via these roads must have at least one road in common. Show that some road must belong to all of the routes from Alphaton to Omegaville.

7. Let $X = \{A_1, A_2, \ldots, A_n\}$ be a set of distinct 3-element subsets of $\{1, 2, \ldots, 36\}$ such that

   (a) $A_i$ and $A_j$ have non-empty intersection for every $i, j$.

   (b) The intersection of all the elements of $X$ is the empty set.

   Show that $n \leq 100$. How many such sets $X$ are there when $n = 100$?

*January 28, 2011*

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