New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2010

Due: 27th August 2010

Junior division

J1. *We number both the rows and the columns of an 8×8 chessboard with the numbers 1 to 8. Some grains of rice are placed on each square, in such a way that the number of grains on each square is equal to the product of the row and column numbers of the square. How many grains of rice are there on the entire chessboard?*

Solution: 1296.

In the first row we have successively 1×1, 1×2, 1×3, $2 \times (1+2+3+\cdots+8)$ …, 1×8 grains of rice, for a total of $1 \times (1+2+3+\cdots+8)$ grains. Similarly, in the second row we have 2×1, 2×2, 2×3, …, 2×8 grains of rice, for a total of $2 \times (1+2+3+\cdots+8)$ grains. In the third row there is a total of $3 \times (1+2+3+\cdots+8)$ grains, and so on, up until the eighth row, which has $8 \times (1+2+3+\cdots+8)$ grains. Adding these up, we see that we have a total of

$$(1+2+3+\cdots+8)^2 = 36^2 = 1296$$

grains of rice. □

J2. *AB is a chord of length 6 in a circle of radius 5 and centre O. A square is inscribed in the sector OAB with two vertices on the circumference and two sides parallel to AB. Find the area of the square.*

Solution: $\frac{900}{109}$.

Label points as in Figure 1, and let the square have sidelength $2a$. Triangle ODZ is similar to the 3-4-5 triangle OEB, so OD has length $4a/3$. Now $OF = OD + DF = 4a/3 + 2a = 10a/3$, and applying the Theorem of Pythagoras to OFY we get $OF^2 = 25 - a^2 = 100a^2/9$. Solving we get $a^2 = (9 \cdot 25)/109$, so the area is $4a^2 = 900/109$. □

J3. *Find all positive integers n such that $n^5 + n + 1$ is prime.*

Solution: The only solution is $n = 1$, for which $1^5 + 1 + 1 = 3$.

Let $f(x) = x^5 + x + 1$. Working out the first few values we have

$$\begin{aligned} f(1) &= 3 &&= 3 \times 1, & f(5) &= 3131 &&= 31 \times 101,\\ f(2) &= 35 &&= 7 \times 5, & f(6) &= 7783 &&= 43 \times 181,\\ f(3) &= 247 &&= 13 \times 19, & f(7) &= 16815 &&= 57 \times 295,\\ f(4) &= 1029 &&= 21 \times 49, & f(8) &= 32777 &&= 73 \times 449. \end{aligned}$$

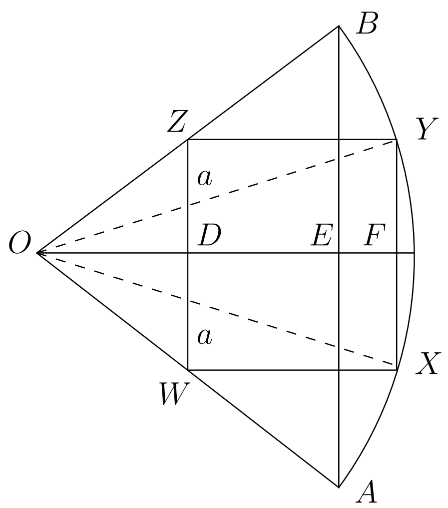

Figure 1: Diagram for problem J2.

Noticing that the left-hand factors above all have the form $n^2 + n + 1$, we are led to suspect that $x^2 + x + 1$ is a factor of $x^5 + x + 1$ (in all but two cases above both factors are prime, so we don't have to make a clever choice of factorisation of $f(n)$ to see these factors). This may be confirmed via long division, which gives

$$x^5 + x + 1 = (x^2 + x + 1)(x^3 - x^2 + 1).$$

For $x > 1$ we have $1 < x^2 + x + 1 < x^5 + x + 1$, so $f(n)$ is not a prime number for $n \geq 2$. Since $f(1) = 3$ is prime this is the only solution.

Remark: Prior to long division it is possible to check whether $x^2 + x + 1$ is a factor using a cube root of unity $\omega \neq 1$. The root ω satisfies $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$ (because $\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1)$), so $\omega^5 = \omega^2$ and we see immediately that $f(\omega) = 0$. □

J4. *Find all positive integer solutions (a, b) to the equation*

$$\frac{1}{a} + \frac{1}{b} + \frac{n}{\operatorname{lcm}(a,b)} = \frac{1}{\gcd(a,b)}$$

for

(i) $n = 2007$;

(ii) $n = 2010$.

Note: *"*$\operatorname{lcm}(a,b)$*" means the least common multiple of a and b, and "*$\gcd(a,b)$*" means the greatest common divisor of a and b. For example,* $\operatorname{lcm}(18,30) = 90$, *and* $\gcd(18,30) = 6$.

Solution: There are no solutions for $n = 2010$, and the solutions are

$$\{(2d, 2009d), (5d, 503d), (503d, 5d), (2009d, 2d) : d \geq 1\}$$

for $n = 2007$.

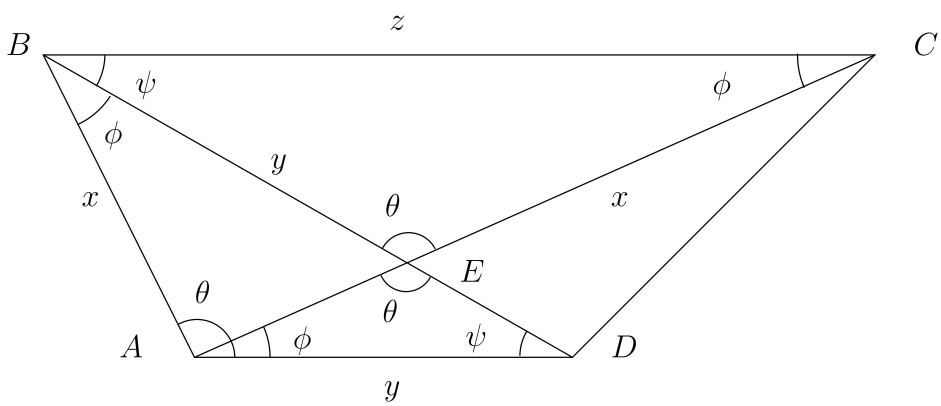

Figure 2: Diagram for problem J5.

Let $d = \gcd(a, b)$. Then there are positive integers x, y which are relatively prime such that $a = dx$, $b = dy$. Using this in the original equation we find $x + y + n = xy$, that is, $n + 1 = (x - 1)(y - 1)$. Since x and y are relatively prime at least one of them is odd, so at least one of $x - 1$, $y - 1$ is even, and so $n + 1$ must be even too. So there is no solution for $n = 2010$.

For $n = 2007$ this gives $(x - 1)(y - 1) = 2008 = 2^3 \cdot 251$, so the possible factorisations are $\{x - 1, y - 1\} = \{1, 2008\}$, $\{2, 1004\}$, $\{4, 502\}$ and $\{8, 251\}$. These give $\{2, 2009\}$, $\{3, 1005\}$, $\{5, 503\}$, and $\{9, 252\}$ as the possibilities for $\{x, y\}$, and of these only the pairs $\{2, 2009\}$ and $\{5, 503\}$ are relatively prime. This gives the list of solutions given above. □

J5. *The diagonals of quadrilateral $ABCD$ intersect in point E. Given that $|AB| = |CE|$, $|BE| = |AD|$, and $\angle AED = \angle BAD$, determine the ratio $|BC|/|AD|$.*

Solution: $\frac{1+\sqrt{5}}{2}$.

See Figure 2. We have $\angle BEC = \angle AED = \angle BAD = \theta$, so triangles CEB and BAD are congruent (side-angle-side). Therefore $\angle BCA = \angle ABD = \phi$, $\angle CBD = \angle BDA = \psi$, and $|BC| = |BD|$. The second equality implies that BC and AD are parallel, and from this or from $\theta + \psi + \phi = \pi$ we have $\angle EAD = \phi$. It follows that triangles AED and CEB are similar.

Let $\lambda = |BC|/|AD|$. Then $|BC| = \lambda|AD| = \lambda|BE| = \lambda^2|ED|$, and $|BC| = |BD| = |BE| + |ED| = (\lambda + 1)|ED|$. Hence λ satisfies the quadratic equation $\lambda^2 = \lambda + 1$, and is therefore the golden ratio $(1 + \sqrt{5})/2$. □

J6. *At a strange party, each person knew exactly 22 others.*

For any pair of people X and Y who knew one another, there was no other person at the party that they both knew.

For any pair of people X and Y who did not know each other, there were exactly six other people that they both knew.

How many people were at the party?

	$x_0 \le 4$, $x_1 \le 4$	$x_0 \le 4$, $x_1 > 4$	$x_0 > 4$, $x_1 \le 4$	$x_0 > 4$, $x_1 > 4$
x_2	$\frac{16}{x_0}$	$\frac{4x_1}{x_0}$	$\frac{16}{x_0}$	$\frac{4x_1}{x_0}$
x_3	$\frac{64}{x_0 x_1}$	$\frac{16}{x_0}$	$\frac{16}{x_1}$	$\max\{\frac{16}{x_0}, \frac{16}{x_1}\}$
x_4	$\frac{16}{x_1}$	$\frac{16}{x_1}$	$\frac{4x_0}{x_1}$	$\frac{4x_0}{x_1}$
x_5	x_0	x_0	x_0	x_0
x_6	x_1	x_1	x_1	x_1

Table 1: Table of values for Problem S1.

Solution: 100.

Suppose there are n people at the party, $p_1, p_2, \ldots, p_n$. Fix i and count the number of distinct ordered pairs (j, k) such that p_i knows p_j and p_j knows p_k. There are 22 pairs where $k = i$. Suppose that $k \neq i$. Then p_k is one of the $n - 22 - 1$ people that p_i doesn't know, and there are 6 people p_j such that we must include (j, k) in our count. So there are $22 + 6(n - 23)$ such pairs.

On the other hand, there must be $22^2 = 484$ such pairs altogether, because each person knows 22 others. Hence $484 = 22 + 6(n - 23)$, and we solve to find $n = 100$. □

Senior division

S1. *For any two positive real numbers $x_0 > 0$, $x_1 > 0$, a sequence of real numbers is defined recursively by*

$$x_{n+1} = \frac{4 \max\{x_n, 4\}}{x_{n-1}} \qquad \textit{for } n \geq 1.$$

Find x_{2010}.

Note: *"$\max\{x, y\}$" means the maximum of x and y — that is, whichever of the two numbers x and y is the larger. For example,* $\max\{2, 3\} = 3$.

Solution: All solutions are periodic with period five, as is easily checked by direct computation (see Table 1; note that we must calculate as far as x_6 to establish the periodicity). Hence $x_{2010} = x_0$. □

S2. *In a convex pentagon $ABCDE$ the areas of the triangles ABC, ABD, ACD and ADE are all equal to the same value x. What is the area of the triangle BCE?*

Solution: $2x$.

We first observe that ABC and ABD have the same base and area, so they must have the same altitude. Hence CD is parallel to AB. We next observe that ABD and ACD have the same altitude and area, so they must have the same base; therefore $|CD| = |AB|$, and $ABCD$ is a parallelogram.

Now observe that ABD and ADE have the same base AD, and the same area, so they too must have the same altitude. In addition, E must be on the opposite side of AD to B, since $ABCDE$ is convex. Considering now the triangles BCD and BCE, we see that they have the same base BC, but the altitude of BCE is twice that of BCD. It follows that it has twice the area, as claimed. $\square$

S3. *Let p be a prime number. Find all pairs (x, y) of positive integers such that*

$$x^3 + y^3 - 3xy = p - 1.$$

Solution: Since we want to use the fact that p is a prime, we add 1 to both sides and try to factorise the left-hand side. Calculating $x^3 + y^3 - 3xy + 1$ for lots of pairs (x, y) gives you the idea that the expression is divisible by $x + y + 1$. It turns out that

$$x^3 + y^3 - 3xy + 1 = (x + y + 1)(x^2 + y^2 - xy - x - y + 1).$$

This needs to be equal to a prime number. Since $x + y + 1 > 1$, we must have $x^2 + y^2 - xy - x - y + 1 = 1$. As $x^2 + y^2 \geq 2xy$, we find

$$1 = x^2 + y^2 - xy - x - y + 1 \geq xy - x - y + 1 = (x - 1)(y - 1).$$

This can only be true if $x = 1$ or $y = 1$ or $x = y = 2$. Suppose $x = 1$, then $2 + y^3 - 3y = p$, but now the left-hand side is even for any y, so we must have $p = 2$. However, that means $y^3 = 3y$, which does not have any integer solutions. Similarly $y = 1$ leads to a contradiction. The only possible solution is $x = y = 2$ and this is a solution if and only if $p = 5$. $\square$

S4. *A line drawn from the vertex A of an equilateral triangle ABC meets the side BC at D and the circumcircle at P. Show that*

$$\frac{1}{|PD|} = \frac{1}{|PB|} + \frac{1}{|PC|}.$$

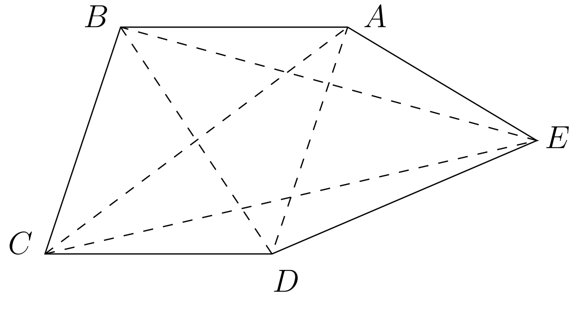

Figure 3: Diagram for problem S2.

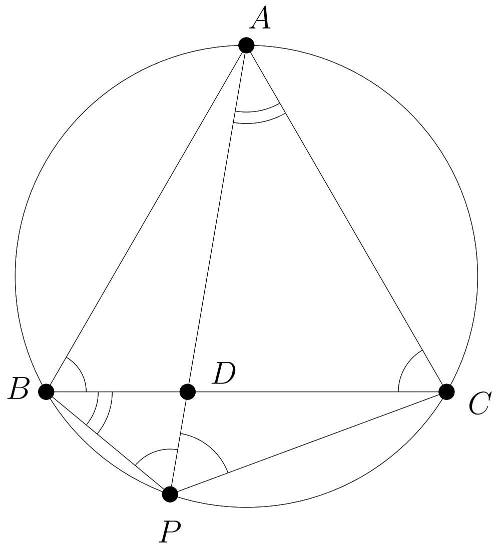

Figure 4: Diagram for problem S4.

Solution: See Figure 4. Because $\angle PAC = \angle PBC$, $\angle APC = \angle ABC = 60^\circ$ and $\angle BPA = \angle BCA = 60^\circ$ the triangles APC and BPD are similar. Thus

$$\frac{|PA|}{|PB|} = \frac{|PC|}{|PD|},$$

so

$$|PA| \cdot |PD| = |PB| \cdot |PC|. \tag{1}$$

Since $ABPC$ is a cyclic quadrilateral, $|PA| \cdot |BC| = |PB| \cdot |AC| + |PC| \cdot |AB|$. But the triangle ABC is equilateral, so it follows that

$$|PA| = |PB| + |PC|. \tag{2}$$

From equations (1) and (2), it follows that

$$|PB| \cdot |PC| = |PD| \cdot (|PB| + |PC|),$$

and we divide by the product $|PB| \cdot |PC| \cdot |PD|$ to get the desired equality

$$\frac{1}{|PD|} = \frac{1}{|PB|} + \frac{1}{|PC|}.$$

□

S5. *Determine the values of the positive integer n for which*

$$A = \sqrt{\frac{9n-1}{n+7}}$$

is rational.

Solution: $n = 1$ or 11.

It is enough to determine for which n there exist positive integers a, b with $\gcd(a, b) = 1$ such that

$$\frac{9n-1}{n+7} = \frac{a^2}{b^2}.$$

From this relation we get

$$n = \frac{7a^2+b^2}{9b^2-a^2} = \frac{7(a^2-9b^2)+64b^2}{9b^2-a^2} = -7 + \frac{64b^2}{9b^2-a^2}.$$

Since $\gcd(a, b) = 1$ it follows that $\gcd(a^2, b^2) = 1$ and $\gcd(9b^2 - a^2, b^2) = 1$, so n is an integer if and only if $9b^2 - a^2$ is a divisor of 64. Moreover $9b^2 - a^2$ must be positive, in order for n to be positive.

Now $9b^2 - a^2 = (3b + a)(3b - a)$. If $a = b = 1$ then $9b^2 - a^2 = 8$; otherwise, $9b^2 - a^2 \geq 3b + a \geq 5$, so $9b^2 - a^2 \geq 8$. So the possible values for $9b^2 - a^2$ are $8, 16, 32, 64$. The factors $3b + a$ and $3b - a$ differ by a multiple of 2, sum to a multiple of 6, and satisfy $3b + a > 3b - a$, so the possibilities for $(3b + a, 3b - a)$ are $(4, 2)$, $(16, 2)$, and $(8, 4)$. The corresponding possibilities for (a, b) are $(1, 1)$, $(7, 3)$ and $(2, 2)$, and we discard $(2, 2)$ because $\gcd(2, 2) = 2 \neq 1$. Substituting the remaining pairs into $n = (7a^2 + b^2)/(9b^2 - a^2)$ we get $n = 1$ or $n = 11$. □

S6. *Suppose a_1, a_2, …, a_8 are eight distinct integers from $\{1, 2, \ldots, 16, 17\}$. Show that there is an integer $k > 0$ such that there are at least three different (not necessarily disjoint) pairs (i, j) such that $a_i - a_j = k$.*

Also find a set of seven distinct integers from $\{1, 2, \ldots, 16, 17\}$ such that there is no integer $k > 0$ with that property.

Solution: Without loss of generality, assume that $a_1 < a_2 < \ldots < a_8$. Consider the numbers $s_1 = a_2 - a_1$, $s_2 = a_3 - a_2$, …, $s_7 = a_8 - a_7$, and $s_8 = a_3 - a_1$, $s_9 = a_4 - a_2$, …, $s_{13} = a_8 - a_6$. We have

$$\sum_{i=1}^{13} s_i = (a_8 - a_1) + (a_8 + a_7 - a_2 - a_1) = 2a_8 + a_7 - a_2 - 2a_1 \leq 2 \cdot 17 + 16 - 2 - 2 \cdot 1 = 46.$$

On the other hand, suppose that for any $k > 0$ there are at most two values of i for which $s_i = k$, then we have

$$\sum_{i=1}^{13} s_i \geq 2 \cdot (1 + 2 + \cdots + 6) + 7 = 49.$$

So this is impossible, and hence there must be a $k > 0$ such that for at least three values of i we have $s_i = k$.

For the second part, take $\{1, 2, 3, 5, 8, 12, 17\}$. □

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