New Zealand Mathematical Olympiad Committee

2010 Squad Assignment Four

Geometry

Due: Wednesday, 14th April 2010

1. A cyclic quadrilateral $ABCD$ is given. Show that the line connecting the orthocentre of the triangle ABC with the orthocentre of the triangle ABD is parallel to the line CD.

2. Let ABC be an isosceles triangle with $|AC| = |BC|$. Its incircle touches AB and BC at D and E respectively. A line (different from AE) passes through A and intersects the incircle at F and G. The lines EF and EG intersect the line AB at K and L, respectively. Prove that $|DK| = |DL|$.

3. Two circles k_a and k_b are given, whose centres lie on the legs of length a and b respectively of a right triangle. Both circles are tangent to the hypotenuse of the triangle and pass through the vertex opposite the hypotenuse.

   Let the circles have radii ρ_a and ρ_b respectively. Find the greatest real number p such that the inequality
$$\frac{1}{\rho_a} + \frac{1}{\rho_b} \geq p\left(\frac{1}{a} + \frac{1}{b}\right)$$
   holds for all right triangles.

4. A line ℓ is given, and on it four points P, Q, R, S (in this order from left to right).

   Construct, with a straight edge and compasses, all squares $ABCD$ satisfying all of the following conditions:

   (a) P lies on the line through A and D;
   (b) Q lies on the line through B and C;
   (c) R lies on the line through A and B; and
   (d) S lies on the line through C and D.

   Note: Be sure to show that your construction works. You may describe it in terms of known constructions: for example, if you need a perpendicular bisector you may simply say "construct the perpendicular bisector to EF", without describing in detail how to do this.

5. Two distinct points O and T are given in the plane ω. Find the locus of vertices of all triangles lying in ω whose centroid is T and whose circumcentre is O.

6. The incircle k of a scalene triangle ABC has centre S and is tangent to the sides BC, CA and AB at the points P, Q and R respectively. The lines QR and BC intersect in the point M. A circle passing through B and C is tangent to k at the point N. The circumcircle of triangle MNP meets the line AP at a point L which is different to P. Prove that the points S, L, M are collinear.

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