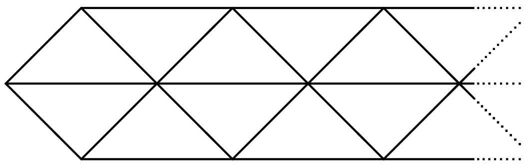
New Zealand Mathematical Olympiad Committee

2010 Squad Assignment Three

Combinatorics

Due: Thursday 18th March 2010

1. In the road network shown below, the vertices in the middle horizontal line are labeled $1, 4, 7, \ldots$, the vertices in the upper row are labelled $2, 5, 8, \ldots$, and the vertices in the bottom row are labelled $3, 6, 9 \ldots$.



   How many paths are there from the vertex labelled 1 to the vertex labelled $3n + 1$ such that vertices are visited only in increasing order?

2. An odd integer is written in each cell of a 2009×2009 table. For $1 \leq i \leq 2009$ let R_i be the sum of the numbers in the ith row, and for $1 \leq j \leq 2009$ let C_j be the sum of the numbers in the jth column. Finally, let A be the product of the R_i, and B the product of the C_j.

   Prove that $A + B$ is different from zero.

3. A number of coins have been placed at each vertex of the regular n-gon $A_1A_2 \ldots A_n$. These coins may be re-arranged using the following move: two coins may be chosen, and each moved to an adjacent vertex, subject to the requirement that one must be moved clockwise and the other anti-clockwise. (Thus, for example, you may move a coin from each of A_1 and A_5 to vertices A_2 and A_4 respectively: each coin ends up on a vertex adjacent to the one it started on, and they move in opposite directions.)

   Suppose that there are initially k coins at vertex A_k for each k, $1 \leq k \leq n$. For which n is it possible to re-arrange the coins using finitely many such moves so that there are exactly $n + 1 - k$ coins at vertex A_k, $1 \leq k \leq n$?

4. A convex 2010-gon is partitioned into triangles using non-intersecting diagonals. One of these diagonals is painted green. The triangulation may be modified using the following move: if ABC and BCD are triangles of the partition having BC as a common side, then the diagonal BC may be replaced by the diagonal AD. Moreover, if BC is green, then it loses its colour and AD becomes green instead. Prove that an arbitrarily chosen diagonal of the polygon can be coloured green using finitely many such operations.

5. Let $n \geq 1$ be an integer. In town X there are n girls and n boys, and each girl knows each boy. In town Y there are n girls, g_1, g_2, ..., g_n, and $2n-1$ boys, b_1, b_2, ..., b_{2n-1}. For $i = 1, 2, \ldots, n$, girl g_i knows boys b_1, b_2, ..., b_{2i-1} and no other boys.

   Let r be an integer with $1 \leq r \leq n$. In each of the towns a party will be held, where r girls from that town are to dance with r boys from the same town in r pairs of dancers. However, each girl will only dance with a boy that she knows. Let $X(r)$ be the number of ways we can choose r pairs of dancers from town X, and let $Y(r)$ be the number of ways that we can choose r pairs of dancers from town Y.

   Show that $X(r) = Y(r)$ for $r = 1, 2, \ldots, n$.

6. Let G be a finite connected graph, whose edges are labelled $1, 2, \ldots, e$ in some order. Starting from an arbitrary vertex, repeat the following process:

   (a) Choose the edge incident to the current vertex with the largest label.

   (b) Move along the chosen edge to the adjacent vertex, relabelling the edge 1, and adding 1 to the labels of all the other edges.

   Prove that eventually each edge is traversed.

7. Determine the largest positive integer n for which there exist pairwise different sets S_1, S_2, ..., S_n with the following properties:

   (a) $|S_i \cup S_j| \leq 2006$ for any two indices $1 \leq i, j \leq n$, and

   (b) $S_i \cup S_j \cup S_k = \{1, 2, \ldots, 2010\}$ for any $1 \leq i < j < k \leq n$.

8. Consider a graph with n vertices, and let k, $1 \leq k \leq n$, be a positive integer. It is known that among any k vertices there exists a vertex which is connected to the remaining $k-1$ vertices. Find all values of n and k for which there must always exist a vertex of degree $n-1$.

4th March 2010

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