New Zealand Mathematical Olympiad Committee

2010 Squad Assignment Two

Algebra

Due: Wednesday, 3rd March 2010

1. *Let x and y be non-negative real numbers. Prove that*
$$(x + y^3)(x^3 + y) \geq 4x^2y^2.$$
When does equality hold?

   Solution: Applying the AM-GM inequality to each factor on the LHS we have
$$\frac{x + y^3}{2}\frac{x^3 + y}{2} \geq \sqrt{xy^3}\sqrt{x^3y} = x^2y^2.$$
The inequality follows.

   For equality we must have equality in each application of the AM-GM, so $x = y^3$ and $y = x^3$. This gives $x = x^9$, with non-negative solutions $x = 0$, $x = 1$. So equality holds if and only if (x, y) equals $(0, 0)$ or $(1, 1)$.

2. *Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the following two conditions:*

   (a) $f(n)$ is a perfect square for all $n \in \mathbb{N}$;

   (b) $f(m + n) = f(m) + f(n) + 2mn$ for all $m, n \in \mathbb{N}$.

   (For the purposes of this problem we will consider $\mathbb{N}$ to be the set $\{1,2,3,\ldots\}$, i.e., 0 is not considered to be a natural number.)

   Solution: By condition (a) we must have $f(1) = a^2$ for some positive integer a. We will prove by induction that $f(n) = na^2 + n(n - 1)$ for all n. The base case $n = 1$ is true, by the definition of a, so suppose that $f(n) = na^2 + n(n - 1)$ for some $n \geq 1$, and consider $f(n + 1)$. By condition (b) with $m = 1$ we have
$$\begin{aligned} f(n + 1) &= f(n) + f(1) + 2n \\ &= na^2 + n(n - 1) + a^2 + 2n \\ &= (n + 1)a^2 + (n + 1)n, \end{aligned}$$
which establishes the inductive step.

   We now aim to show that $a = 1$. Suppose then that $a > 1$, and let p be a prime factor of a. Then $f(p) = pa^2 + p(p - 1)$, and since it is divisible by p, it must be divisible by p^2. Hence $a^2 + p - 1$ is divisible by p. But this is a contradiction, as a and p are both divisible by p, so a must equal 1.

   Therefore $f(n) = n^2$ for all n is the only remaining possibility for f. This does indeed satisfy the conditions, and so is the only solution.

3. *Consider two ordered sequences of real numbers* $a_1 < a_2 < \cdots < a_n$ *and* $b_1 < b_2 < \cdots < b_m$*, where* $n, m \in \mathbb{N}$, $n, m \geq 2$*. Prove that the set*

$$\{a_i + b_j | 1 \leq i \leq n, 1 \leq j \leq m\}$$

contains exactly $n+m-1$ *elements if and only if both sequences are arithmetic, with the same increment.*

Solution: Suppose first that the sequences are both arithmetic, with the same increment d. Then $a_i = a_1 + (i-1)d$ and $b_j = b_1 + (j-1)d$, giving $a_i + b_j = a_1 + b_1 + (i + j - 2)$. This gives a total of $n + m - 1$ distinct values.

To prove the opposite direction, place the sum $a_i + b_j$ at the point (i, j) in the plane, and consider paths from $(0, 0)$ to (n, m) that take steps of either $(0, 1)$ or $(1, 0)$ at each stage. Any such path requires exactly $n + m - 2$ steps and visits $n + m - 1$ sums; moreover, the sums visited increase monotonically along the path, because $a_i + b_j < a_{i+1} + b_j$ and $a_i + b_j < a_i + b_{j+1}$. The $n + m - 1$ sums visited along any such path must therefore be distinct; and since there are exactly $n + m - 1$ distinct sums, every such path visits all $n + m - 1$ distinct sums, in increasing order.

The previous paragraph implies that $a_i + b_j = a_k + b_l$ whenever $i + j = k + l$ (consider the $(i + j + 1)$th sum visited along a path that begins with i steps left, followed by j steps up). For $1 \leq i \leq n-1$, $1 \leq j \leq m-1$ we therefore have

$$a_{i+1} + b_j = a_i + b_{j+1},$$

or

$$a_{i+1} - a_i = b_{j+1} - b_j.$$

The result follows.

Alternate solution. The reverse implication may also be proved by induction on $n + m$, as follows. The base case is $n + m = 4$ (when $n = m = 2$), and here we have $a_1 + b_1 < a_2 + b_1 < a_2 + b_2$, $a_1 + b_1 < a_1 + b_2 < a_2 + b_2$. Since there are only three distinct sums we must have $a_1 + b_2 = a_2 + b_1$, from which we obtain $a_2 - a_1 = b_2 - b_1$.

For the inductive step, we begin by observing (similarly to above) that $a_1 + b_1 < a_2 + b_1 < \cdots < a_n + b_1 < a_n + b_2 < \cdots < a_n + b_m$, so there are necessarily at least $n + m - 1$ distinct values. Suppose now that $n + m > 4$, and without loss of generality assume that $m > 2$. Consider the set of sums obtained by eliminating b_m from the sequence $\{b_i\}$. This eliminates at least the sum $a_n + b_m$, which is maximal, so it contains at most $n + m - 2$ values; but by the observation above it must also contain at least $n + m - 2$ values. It therefore contains exactly $n + m - 2$ values, and by the induction hypothesis, the sequences $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_{m-1}$ are arithmetical with the same increment.

We now apply the same argument to the sequences obtained by eliminating b_1. This removes at least the sum $a_1 + b_1$, which is minimal, and as in the previous paragraph we conclude that the sequences $a_1 < a_2 < \cdots < a_n$ and $b_2 < b_3 < \cdots < b_m$ are arithmetical with the same increment. The result now follows.

4. *Determine the largest subset $M \subseteq \mathbb{R}^+$ such that the inequality*

$$\sqrt{ab} + \sqrt{cd} \geq \sqrt{a+b} + \sqrt{c+d}$$

holds for all $a, b, c, d \in M$.

Determine whether the inequality

$$\sqrt{ab} + \sqrt{cd} \geq \sqrt{a+c} + \sqrt{b+d}$$

also holds for all $a, b, c, d \in M$. (Note that $\mathbb{R}^+$ denotes the set of all positive real numbers.)

Solution: If the inequality is to hold for all a, b, c, d in M, it must certainly hold when $a = b = c = d \in M$. In this case the inequality becomes $2a \geq 2\sqrt{2a}$, which is equivalent to

$$a^2 \geq 2a \quad \Leftrightarrow a^2 - 2a \geq 0 \quad \Leftrightarrow a(a-2) \geq 0.$$

For $a > 0$ this is equivalent to $a \geq 2$, and therefore $M \subseteq [2, \infty)$.

If $a, b \in [2, \infty)$ then $a - 1, b - 1 \geq 1$, so $(a-1)(b-1) \geq 1$. This gives $ab \geq a + b$, and taking the square root we get $\sqrt{ab} \geq \sqrt{a+b}$. Since the same holds for c and d in $[2, \infty)$ it follows that

$$\sqrt{ab} + \sqrt{cd} \geq \sqrt{a+b} + \sqrt{c+d}$$

for all $a, b, c, d \in [2, \infty)$. Therefore $M = [2, \infty)$.

We now consider the second inequality $\sqrt{ab}+\sqrt{cd} \geq \sqrt{a+c}+\sqrt{b+d}$, which is equivalent to

$$ab + cd + 2\sqrt{abcd} \geq a + b + c + d + 2\sqrt{(a+c)(b+d)}.$$

Since we already know that $ab \geq a + b$ and $cd \geq c + d$ for $a, b, c, d \in M$, we need only show that $\sqrt{abcd} \geq \sqrt{(a+c)(b+d)}$. This is equivalent to $(ac)(bd) \geq (a+c)(b+d)$, and since we have $ac \geq a + c$, $bd \geq b + d$ for $a, b, c, d \in M$, we see that the second inequality holds also.

5. *Let a, b, c be positive real numbers such that $a + b + c \geq abc$. Prove that*

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc.$$

Solution: First of all note that $a^2 + b^2 + c^2 \geq ab + bc + ca$, by the rearrangement inequality, and hence $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$. By the AM-GM inequality we have $a + b + c \geq 3(abc)^{1/3}$, and by the given condition we have $a + b + c \geq abc$; taking each of these together with the inequality $3(a^2+b^2+c^2) \geq (a+b+c)^2$ we get the two inequalities

$$a^2 + b^2 + c^2 \geq \frac{1}{3}(a+b+c)^2 \geq 3(abc)^{2/3}$$
$$a^2 + b^2 + c^2 \geq \frac{1}{3}(a+b+c)^2 \geq \frac{1}{3}(abc)^2.$$

Since everything is positive we may raise the first of these to the 3/4-th power, and the second to the 1/4-th, to get

$$a^2 + b^2 + c^2 \geq 3^{3/4}(abc)^{1/2}$$
$$a^2 + b^2 + c^2 \geq 3^{-1/4}(abc)^{1/2}.$$

Taking the product of these two expressions gives the desired result.

6. *Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying*

$$f(m+n) + f(mn-1) = f(m)f(n) + 2$$

for all $m, n \in \mathbb{Z}$.

Solution: Suppose there exists $c \in \mathbb{Z}$ such that $f(n) = c$ for all n. Then $2c = c^2 + 2$, so c is a root of the quadratic $c^2 - 2c + 2 = (c-1)^2 + 1 = 0$. This has no real solutions, so f cannot be constant.

Now substitute $m = 0$. This yields the equation $f(n) + f(-1) = f(0)f(n) + 2$, from which we deduce that $f(n)(1 - f(0))$ is constant. Since $f(n)$ is not constant, we must have $f(0) = 1$. The same equation then gives $f(-1) = 2$. We now substitute $m = n = -1$ to get $f(-2) + f(0) = f(-1)^2 + 2$, from which we obtain $f(-2) = 5$; and then substitute $m = 1$, $n = -1$, to get $f(0) + f(-2) = f(1)f(-1) + 2$, giving $f(1) = 2$.

Now that we have $f(1) = 2$ we may substitute $m = 1$ to get $f(n+1) + f(n-1) = 2f(n) + 2$, or equivalently,

$$f(n+1) = 2f(n) - f(n-1) + 2.$$

This is a nonhomogeneous second order linear recurrence relation, which may be solved for positive n using standard techniques (see for example *Recurrence Relations*, on the NZMOC site), or the solution $f(n) = n^2 + 1$ may be guessed and checked for $n \geq 1$ by induction. A second induction using

$$f(n-1) = 2f(n) - f(n+1) + 2$$

shows that $f(n) = n^2 + 1$ for $n < 0$ also (alternately, it is possible to show that f must be even).

We've now shown that if the functional equation has a solution at all, it must be $f(n) = n^2 + 1$. It remains to check that this function is in fact a solution — afterall, in our work above we always had m or n equal to 0 or 1, so we need to make sure the functional equation is satisfied for all possible values of m and n. This is easily done by substitution.

7. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers such that $a_n < a_{n+1}$ for all $n \geq 1$. Suppose that for all 4-tuples of indices (i, j, k, l) such that $1 \leq i < j \leq k < l$ and $i + l = j + k$, the inequality $a_i + a_l > a_j + a_k$ is satisfied. Determined the least possible value of a_{2010}.*

Solution: Applying the inequality to the 4-tuple $(i, j, k, l) = (n-1, n, n, n+1)$ we obtain

$$a_{n-1} + a_{n+1} > 2a_n,$$

or

$$a_{n+1} - a_n > a_n - a_{n-1}.$$

Hence $a_{n+1} - a_n \geq a_n - a_{n-1} + 1$. Now $a_2 > a_1$, so $a_2 - a_1 \geq 1$, and induction gives $a_n \geq a_{n-1} + n$. A second induction using $a_1 \geq 1$ gives

$$a_n \geq \frac{n(n+1)}{2} + 1.$$

We now show that the sequence $a_n = n(n+1)/2+1$ satisfies the conditions in the problem. The inequality $a_i + a_l > a_j + a_k$ is equivalent to

$$(i^2 + i + 2) + (l^2 + l + 2) > (j^2 + j + 2) + (k^2 + k + 2),$$

and when $i+l = j+k$ this is simply $i^2 + l^2 > j^2 + k^2$. Let $i = d - y$, $l = d + y$, $j = d - x$, $k = d + x$, with $0 \geq x < y$. Then $i^2 + l^2 = 2(d^2 + y^2)$, $j^2 + k^2 = 2(d^2 + x^2)$, and the desired inequality follows from $y^2 > x^2$.

The smallest possible value of a_{2010} is therefore

$$\frac{2010 \cdot 2009}{2} + 1 = 2,019,046.$$

8. *Find the least positive number x with the following property: if a, b, c, d are arbitrary positive numbers whose product is 1, then*

$$a^x + b^x + c^x + d^x \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

Solution: Putting the righthand side over the common denominator $abcd$, and using the condition that $abcd = 1$, we need to find the smallest x such that

$$a^x + b^x + c^x + d^x \geq bcd + acd + abd + abc.$$

The righthand side is now degree 3 in a, b, c, d; for the lefthand side to surpass this we might expect that it must have degree at least 3 as well, or else the inequality will fail when the variables take large values. This suggests that the answer might be $x = 3$, and we will show that this is indeed the case.

We first show that $x = 3$ works. Applying the AM-GM inequality to the numbers a^3, b^3, c^3 we obtain

$$\frac{a^3 + b^3 + c^3}{3} \geq \sqrt[3]{a^3 b^3 c^3} = abc;$$

similarly

$$\frac{a^3 + b^3 + d^3}{3} \geq abd, \qquad \frac{a^3 + c^3 + d^3}{3} \geq acd, \qquad \frac{a^3 + b^3 + d^3}{3} \geq abd.$$

Adding these four inequalities we obtain

$$a^3 + b^3 + c^3 + d^3 \geq bcd + acd + abd + abc,$$

as desired.

We now show that if $x < 3$ then we may find (a, b, c, d) such that $abcd = 1$ and the inequality is violated. To do this we will try to realise our intuition above that the inequality will fail when the variables are large. We may only choose three of the variables independently, so we will look for a quadruple of the form $a = b = c = t$, $d = 1/t^3$, with $t > 1$. For such quadruples we certainly have $abcd = 1$, while

$$a^x + b^x + c^x + d^x = 3t^x + \frac{1}{t^{3x}} < 4t^x, \qquad \text{and} \qquad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{3}{t} + t^3 > t^3.$$

Thus, if we can choose t such that $4t^x < t^3$, then the desired inequality will be violated. In view of the condition $x < 3$, the inequality $4t^x < t^3$ is equivalent to

$$t > 4^{\frac{1}{3-x}},$$

which is certainly fulfilled for t large enough.

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