New Zealand Mathematical Olympiad Committee

2010 Squad Assignment Two

Algebra

Due: Wednesday, 3rd March 2010

1. Let x and y be non-negative real numbers. Prove that

$$(x + y^3)(x^3 + y) \geq 4x^2y^2.$$

   When does equality hold?

2. Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the following two conditions:

   (a) $f(n)$ is a perfect square for all $n \in \mathbb{N}$;

   (b) $f(m + n) = f(m) + f(n) + 2mn$ for all $m, n \in \mathbb{N}$.

   (For the purposes of this problem we will consider $\mathbb{N}$ to be the set $\{1,2,3,\ldots\}$, i.e., 0 is not considered to be a natural number.)

3. Consider two ordered sequences of real numbers $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_m$, where $n, m \in \mathbb{N}$, $n, m \geq 2$. Prove that the set

$$\{a_i + b_j | 1 \leq i \leq n, 1 \leq j \leq m\}$$

   contains exactly $n + m - 1$ elements if and only if both sequences are arithmetic, with the same increment.

4. Determine the largest subset $M \subseteq \mathbb{R}^+$ such that the inequality

$$\sqrt{ab} + \sqrt{cd} \geq \sqrt{a+b} + \sqrt{c+d}$$

   holds for all $a, b, c, d \in M$.

   Determine whether the inequality

$$\sqrt{ab} + \sqrt{cd} \geq \sqrt{a+c} + \sqrt{b+d}$$

   also holds for all $a, b, c, d \in M$. (Note that $\mathbb{R}^+$ denotes the set of all positive real numbers.)

5. Let a, b, c be positive real numbers such that $a + b + c \geq abc$. Prove that

$$a^2 + b^2 + c^2 \geq \sqrt{3}abc.$$

6. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(m+n) + f(mn-1) = f(m)f(n) + 2$$

for all $m, n \in \mathbb{Z}$.

7. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive integers such that $a_n < a_{n+1}$ for all $n \geq 1$. Suppose that for all 4-tuples of indices (i, j, k, l) such that $1 \leq i < j \leq k < l$ and $i + l = j + k$, the inequality $a_i + a_l > a_j + a_k$ is satisfied. Determined the least possible value of a_{2010}.

8. Find the least positive number x with the following property: if a, b, c, d are arbitrary positive numbers whose product is 1, then

$$a^x + b^x + c^x + d^x \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

17th February 2010

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