New Zealand Maths Olympiad Committee
September Problems 2009
Due: October 21

Junior Division

1. For which values, if any, of the positive integer n is $n(4n+1)$ a perfect square?

2. A square $ABCD$ with sides of length 1 is labeled with A in the lower left corner, and proceeding counter-clockwise. A horizontal line segment EW (E at the left), and a vertical line segment NS (N at the top), both of length 1/2 lie entirely inside the square, and intersect at a point X. What is the sum of the areas of the triangles EDN, SEA, WSB and NWC?

3. Let A be a subset of $\{1, 2, 3, \ldots, 2010\}$ having the property that the difference of any two elements of A is not a prime number. What is the largest possible number of elements of A? (Note, 1 is not a prime number).

4. In $\triangle ABC$, $\angle CAB = 2\angle ABC$. Let the side lengths of BC, CA and AB be a, b and c respectively. Prove that $a^2 = b(b+c)$.

5. Let x and y be two integers such that $x^2 + 2y$ is a perfect square. Prove that $x^2 + y$ is a sum of two perfect squares.

6. At a certain math camp there were four times as many boys as girls (sad, but true). One day, all the students sat down around a circular table. One of the adults noticed that among the pairs of students sitting next to each other there were three times as many pairs of the same sex as there were pairs of opposite sexes. What is the smallest possible number of students who were attending the camp?

Senior Division

1. Let k be a positive integer. Show that the number of powers of 2 that have k digits in decimal notation is at least three and at most four. Given that the largest power of 2 which is less than 10^{2009} is 2^{6673}, for how many k between 1 and 2009 inclusive, are there four powers of 2 that have k digits?

2. Let a, b, c and d be integers. Show that the equation

$$x^2 + ax + b = y^2 + cy + d$$

has infinitely many integer solutions (x, y) if and only if $a^2 - 4b = c^2 - 4d$.

3. Let A, B, C, D, E, F and G be seven consecutive vertices of a regular dodecagon. Segments AF and BG intersect at point P, lines AC and GE meet at point Q, and lines AB and GF meet at point R. Show that P is the orthocentre, and Q is the circumcentre, of triangle ARG. [A triangle's *orthocentre* is the point of intersection of its altitudes, and its *circumcentre* is the centre of the circumscribed circle.]

4. Let a number $x \neq 0, 1$, and a positive integer n be given. A sequence of numbers a_0, a_1, a_2, $\ldots$, a_n is defined by: $a_0 = x$, $a_1 = 1 - x$, and $a_k = 1 - a_{k-1}(1 - a_{k-1})$ for $k = 2, 3, \ldots, n$. Prove that:

$$a_0 a_1 \cdots a_n \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right) = 1.$$

5. Let ABC be a triangle, and let Γ be its circumcircle. Suppose that a circle with centre O is tangent to the segment BC at a point P, and to the circle Γ at Q which lies in the arc of Γ determined by BC not containing A. Finally suppose that $\angle BAO = \angle CAO$. Prove that $\angle PAO = \angle QAO$.

6. There were 20 players at a chess tournament. Each player played every other player exactly once, and either one of them won, or the game was a draw. It so happened that if a game ended in a draw, then each of the other 18 players beat at least one of the two players involved in the draw. At least two games ended in a draw. Show that it is possible to order the players as P_1, P_2, $\ldots$, P_{20} so that P_k beat P_{k+1} for each k from 1 to 19 inclusive.