New Zealand Mathematical Olympiad Committee

Limits, continuity and completeness

Heather Macbeth

1 Introduction

Real analysis is the study of sequences and functions of real numbers.

What is special about the real numbers? Intuitively and imprecisely, the real numbers are what one gets by taking the rational numbers and 'filling in the gaps.' To formalize this notion, let us state the following theorem:

Theorem 1. *There exists a complete Archimedean field. Moreover, any two complete Archimedean fields are isomorphic.*

and make the following definition:

Definition. The *real numbers* are the unique (up to isomorphism) complete Archimedean field.

The rest of these notes will outline what it means to be a complete Archimedean field, and explore why these notions are useful.

2 Limits

Archimidean fields

The rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ are both examples of Archimedean fields. So is, for instance, the *adjunction of* $\sqrt[3]{5}$ *to* $\mathbb{Q}$; that is, the set of all real numbers of the form $a + b\sqrt[3]{5} + c\sqrt[3]{25}$, where a, b and c are rational. (Though we won't define what we mean by by $\sqrt[3]{5}$ until later on, in Example 8!)

Roughly speaking, an Archimidean field is a set of numbers which one can add, subtract, multiply and divide without leaving the set, and which are ordered in a sensible way. One consequence is that any Archimedean field contains, at the very least, the rational numbers.

Archimedean fields are a natural setting in which to define the basic concepts of analysis, and that's what we'll do for the rest of this section. For a more precise statement of the Archimedean field axioms, refer to any proper introduction to analysis (or to Wikipedia).

Basic facts about limits

Let $\mathbb{K}$ be an Archimidean field. Let $x_1, x_2, x_3, \ldots$ be a sequence of numbers in $\mathbb{K}$, and let x be a number in $\mathbb{K}$.

Definition. The number x is a *limit* of the sequence $x_1, x_2, x_3, \ldots$, if for all positive numbers $\epsilon > 0$ in $\mathbb{K}$, there is some number N in $\mathbb{K}$, such that for all natural numbers $n \geq N$,

$$|x_n - x| < \epsilon.$$

A sequence is *convergent* in if it has a limit in $\mathbb{K}$, and *divergent* if it has no limit.

Here are some facts that can be easily proved from that definition, which show that limits behave as one would expect.

Theorem 2. *If a sequence has a limit, then that limit is unique.*

Hence it makes sense to talk about *the* limit of a convergent sequence of numbers in $\mathbb{K}$. The limit of a convergent sequence $x_1, x_2, x_3, \dots$ is denoted $\lim_{n\to\infty} x_n$.

Theorem 3. *In any Archimedean field $\mathbb{K}$, for any $x \in \mathbb{K}$, there is a sequence of rational numbers with limit x.*

Theorem 4. *If x and y are limits respectively of the two sequences $x_1, x_2, x_3, \dots$ and $y_1, y_2, y_3, \dots$, then $x + y$ is a limit of their element-wise sum $x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots$.*

The analogous statement holds for products.

First Cardinal Rule of Analysis. *Never make use of the limit of a sequence, without first proving that that limit exists.*

What use is the notion of 'limit'?

The notion of 'limit' is useful because it allows all sorts of other fundamental concepts to be formulated.

Example 1. A function $f : \mathbb{K} \to \mathbb{K}$ is *continuous*, if for all convergent sequences $x_1, x_2, x_3, \dots$ in $\mathbb{K}$, the sequence $f(x_1), f(x_2), f(x_3), \dots$ is convergent and has limit $f(\lim_{n\to\infty} x_n)$.

Example 2. Let x and y be numbers in $\mathbb{K}$. A function $f : \mathbb{K} \to \mathbb{K}$ is said to *have derivative y at x*, if for all sequences $x_1, x_2, x_3, \dots$ in $\mathbb{K}$ which have limit x and don't contain x, the sequence

$$\frac{f(x_1) - f(x)}{x_1 - x}, \ \frac{f(x_2) - f(x)}{x_2 - x}, \ \frac{f(x_3) - f(x)}{x_3 - x}, \ \dots$$

is convergent and has limit y.

Second Cardinal Rule of Analysis. *Never make use of the derivative of a function, without first proving that that derivative exists.*

Incidentally, the 'don't contain x' part of that last definition illustrates the

Zeroth Cardinal Rule of Analysis. *Never divide by (or cancel) a number, without first proving that that number is nonzero.*

3 Completeness

The property which makes the real numbers unique among Archimedean fields is *completeness.* We will give four equivalent defintions of this property.

Exercise 1. Prove that these four definitions are equivalent.

Nested intervals

A *nested sequence of intervals* in $\mathbb{K}$ is a sequence $[a_1, b_1],\ [a_2, b_2],\ [a_3, b_3],\ \dots$ of intervals with endpoints in $\mathbb{K}$, each containing the last. That is, for all n, we have $a_n < b_n$, and moreover

$$a_1 \le a_2 \le a_3 \le \cdots \quad \text{and} \quad b_1 \ge b_2 \ge b_3 \ge \cdots .$$

Definition. A *complete* Archimedean field $\mathbb{K}$ is an Archimedean field such that, for each nested sequence $[a_1, b_1],\ [a_2, b_2],\ [a_3, b_3],\ \dots$ of intervals in $\mathbb{K}$, some number $x \in \mathbb{K}$ is contained in all of these intervals.

Example 3. Construct a nested sequence of intervals with rational endpoints recursively as follows: $[a_1, b_1] = [1, 2]$, and, given the interval $[a_k, b_k]$, we define

$$[a_{k+1}, b_{k+1}] = \begin{cases} [a_k, m], & \text{if } m^2 > 2 \\ [m, b_k], & \text{if } m^2 < 2, \end{cases}$$

where $m = (a_k + b_k)/2$. So $[a_2, b_2] = [1, 1.5]$, $[a_3, b_3] = [1.25, 1.5]$, etc. No rational number is contained in all of these intervals. (Although any positive number with square equal to 2, would be!) Hence the rational numbers are not complete.

Bounded monotonic sequences

A sequence $x_1, x_2, x_3, \ldots$ is *nondecreasing*, if $x_1 \leq x_2 \leq x_3 \leq \cdots$. It is *nonincreasing*, if $x_1 \geq x_2 \geq x_3 \geq \cdots$. It is *monotonic*, if it is either nondecreasing or nonincreasing.

A subset $X \subseteq K$ of an Archimedean field is *bounded above* (opposite: *bounded below*), if there is some number M in $\mathbb{K}$, such that for all numbers $x \in X$, we have $x < M$. It is *bounded*, if it is bounded both above and below.

Definition. A *complete* Archimedean field is an Archimedean field in which every bounded monotonic sequence converges. (Or, equivalently, in which every nondecreasing bounded-above sequence converges.)

Example 4. The sequence $a_1, a_2, a_3, \ldots$ of lower endpoints to the intervals constructed in Example 3 is a bounded, monotonic sequence of rational numbers, but has no rational limit. Hence the rational numbers are not complete.

Suprema and infima

A *supremum* (opposite: *infimum*) of a subset X of an Archimedean field $\mathbb{K}$, is a number $\xi \in \mathbb{K}$ with the following two properties:

(i) For all $x \in X$, we have $x \leq \xi$.

(ii) There is a sequence $x_1, x_2, x_3, \ldots$ of numbers in X, such that

$$\lim_{n \to \infty} x_n = \xi.$$

Definition. A *complete* Archimedean field is an Archimedean field in which every nonempty, bounded-above set of numbers has a supremum.

Example 5. The set of rational numbers whose squares are less than 2 is bounded and nonempty, but has no rational supremum. Hence the rational numbers are not complete.

By contrast, a *maximum* (opposite: *minimum*) of a subset X of an Archimedean field $\mathbb{K}$, is a number $\xi \in \mathbb{K}$ with the following two properties:

(i) For all $x \in X$, we have $x \leq \xi$.

(ii) The number ξ is actually contained in the set X. That is, $\xi \in X$.

A set can have at most one supremum, and at most one maximum. If a set has a maximum, that maximum is also its supremum; on the other hand, a set may well have a supremum but no maximum. Completeness guarantees the existence of suprema, but certainly not the existence of maxima.

Example 6. The set of negative numbers is bounded-above and nonempty. It has supremum 0, but no maximum.

Third Cardinal Rule of Analysis. *Never make use of the maximum of a set, without first proving that that maximum exists.*

Exercise 2. Let $X \subseteq \mathbb{R}$ be a nonempty bounded-below subset of the reals. Let Y be the set of all *lower bounds* for X; that is, the set of all real numbers $M \in \mathbb{R}$, such that for all $x \in X$, we have $x \geq M$. Show that Y has a maximum, and that that maximum is the infimum of X.

Hence, despite the above warning, it's permissible to make use of the maximum of a set of lower bounds: that maximum must always exist. (Likewise, the minimum of a set of upper bounds must exist.)

Cauchy sequences

A sequence $x_1, x_2, x_3, \ldots$ is *Cauchy*, if for each positive number $\epsilon > 0$, there exists some number N in $\mathbb{K}$, such that for all natural numbers $n, m \geq N$, it is true that

$$|x_n - x_m| < \epsilon.$$

Definition. A *complete* Archimedean field is an Archimedean field in which every Cauchy sequence converges.

Example 7. The sequence $a_1, a_2, a_3, \ldots$ of lower endpoints to the intervals constructed in Example 3 is a Cauchy sequence of rational numbers, but has no rational limit. Hence the rational numbers are not complete.

What use is the notion of completeness?

Completeness – the formalization of that idea that the real numbers 'fill in the gaps' between the rational numbers – is the axiom that uniquely defines the real numbers. It is therefore the source of all the properties that make the real numbers special. Try using it to prove some of the following.

Example 8. There exists a positive real number whose square is 2. There exists a positive real number whose cube is 5. (We refer to these as $\sqrt{2}$ and $\sqrt[3]{5}$, respectively.)

Example 9 (Intermediate Value Theorem)**.** Let $f : [a_1, a_2] \to \mathbb{R}$ be continuous, where $[a_1, a_2] \subseteq \mathbb{R}$ is a real interval. Suppose $f(a_1) < b$ and $f(a_2) > b$. Then there is some real number $x \in [a_1, a_2]$, such that $f(x) = b$.

Example 10 (Weierstrass' Theorem)**.** Let $f : [a_1, a_2] \to \mathbb{R}$ be continuous, where $[a_1, a_2] \subseteq \mathbb{R}$ is a real interval. Then the set $\{f(x) : x \in [a_1, a_2]\}$ has a maximum.

4 Problems

1. Let $\mathbb{K}$ be an Archimedean field. Find all continuous functions $f : \mathbb{K} \to \mathbb{K}$, such that for all $x, y \in \mathbb{K}$,

$$f(x + y) = f(x) + f(y).$$

2. (a) Let $\mathbb{K}$ be an Archimedean field, and suppose that $f : \mathbb{K} \to \mathbb{K}$ is strictly increasing. Show that if f is bijective then it is continuous.

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing. Show that if f is continuous and bounded neither above nor below, then it is bijective.

3. Denote by $\mathbb{R}^+$ the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^+$,

$$f(x + f(y)) = x + f(y).$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that for each $x \in \mathbb{R}$, some term of the sequence

$$f(x),\ f(f(x)),\ f(f(f(x))),\ \ldots$$

is equal to 1. Show that $f(1) = 1$.

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