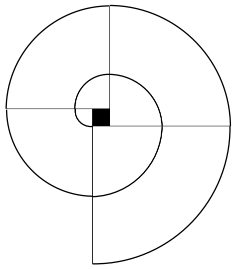
New Zealand Mathematical Olympiad Commitee

January Solutions

1. *Rex the dog is attached by an 8 m long chain to the south-west corner of his 1m square doghouse (whose sides run in the four primary compass directions.) A mischievous cat leads him, at full stretch of the chain at all times, for two circuits around the doghouse, beginning from a point directly south of the south-west corner. How much farther does he travel in the first lap than in the second?*

   Solution:

   Rex's run can be divided up as seen above into eight quarter-circles, the first of radius 8, the second of radius 7, and so on. So the first lap has length $(\pi/2)(8+7+6+5)$ and the second $(\pi/2)(4+3+2+1)$, and the difference is $(\pi/2)(4+4+4+4) = 8\pi$.

2. *In a 3×3 magic square, the three row sums, the three column sums, and the two diagonal sums must be the same. Given the partially completed square below, what is the value of D?*

A	B	6
D	E	F
G	9	2

   Solution: From $6+E+G = G+9+2$, we get $E = 5$. Then from $D+E+F = 6+F+2$ we get $D+5 = 8$ and hence $D = 3$.

3. *How many pairs (a, b) are there where a and b are integers in the range from 1 to 100 inclusive, and a^b is a perfect square.*

   Solution: If $b = 2c$ is even, then for any a, $a^b = a^{2c} = (a^c)^2$, so all 100×50 pairs of this type are suitable. However, if $b = 2c + 1$ is odd, then $a^b = (a^c)^2 \times a$ and this is a perfect square if and only if a is. Hence, only 10×50 pairs of this type are suitable. The total number of suitable pairs is thus $110 \times 50 = 5500$.

4. *Is it possible to partition the set $\{1, 2, 3, \ldots, 45\}$ into nine 5-element subsets (this means that every number from the set belongs to exactly one of the subsets) in such a way that in each of the subsets we can find three elements whose sum is equal to the sum of the other two?*

   Solution: Suppose that this were possible and let $\{a, b, c, d, e\}$ be a typical one of the sets with (say) $a + b + c = d + e$. Then, $a + b + c + d + e = 2d + 2e$ and in particular, the sum of each of the subsets is even. Thus the sum of the sums of each set, i.e. the sum of all the numbers from 1 through 45 would have to be even. But, this sum is $(45 \times 46)/2 = 45 \times 23$ which is odd. So, no such partition is possible.

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