New Zealand Maths Olympiad Committee
2009 Maths Gymnastics
Christchurch, Thursday 14 January

1. Simplify
$$\frac{\sin^4 x + \cos^4 x - 1}{\sin^6 x + \cos^6 x - 1}$$

2. Find all integers $n \geq 2$ such that
$$\frac{3n^2 - 3n + 20}{n - 1}$$
is a positive integer.

3. Find the value of
$$\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ}$$

4. Find all real numbers x such that
$$\sqrt[4]{8 - x} + \sqrt[4]{89 + x} = 5.$$

5. Find the value of $\tan \frac{x}{2}$, if $\sin x - \cos x = 1.4$.

6. Solve the following inequality
$$\frac{\sin 2x - \cos 2x + 1}{\sin 2x + \cos 2x + 1} > 0.$$

7. What is the value of $8 \cos \frac{4\pi}{9} \cos \frac{2\pi}{9} \cos \frac{\pi}{9}$?

8. I am thinking of a positive integer x.

   - When you divide x by 2, the remainder is 1.
   - When you divide x by 3, the remainder is 2.
   - When you divide x by 4, the remainder is 3.
   - When you divide x by 5, the remainder is 4.

   What are the possible values for my integer x?

9. Simplify: $\cos x + \cos 2x + \cos 3x + \cdots + \cos 100x$

10. The longest median of a right-angled triangle has length 8. What is the length of the hypotenuse?

11. Prove that for all triplets (a, b, c) of positive real numbers,

$$ab(a + b - 2c) + bc(b + c - 2a) + ac(a + c - 2b) \geq 0.$$

12. A cyclic quadrilateral has side lengths 1, 5, 5 and 7. What is its area?

13. Suppose

$$\tan \frac{x}{2} = \frac{a}{b}$$

   Prove that $a \sin x + b \cos x$ does not depend on x.