New Zealand Maths Olympiad Committee
September Problems 2008
Due: 25 October

Junior Division

1. When amoebas of a particular exotic species reproduce, the parent amoeba splits apart into four identical baby amoebas. Over time, the members of a colony of these amoebas occasionally reproduce, and the colony's size grows. Suppose the colony contains four amoebas to start with. Can its number of members ever reach exactly 2009?

2. An isosceles triangle is inscribed in a circle, so that one of the sides of the triangle is the circle's diameter. What are the angles of the triangle?

3. Show that
$$n\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) \geq (n+1)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$$
for all natural numbers $n \geq 1$.

4. A right-angled triangle has legs of length a and b. A circle of radius r touches the two legs and has its centre on the hypotenuse. Show that $\frac{1}{a}+\frac{1}{b}=\frac{1}{r}$.

5. Michael plans to buy a set of weights for his balance scale, so that the following conditions are satisfied:

   (a) The set contains weights of at least five different masses.

   (b) For any two weights in the set, two more weights from the set can be picked, so that the sums of the masses of the two pairs of weights are the same.

   He moreover wants to buy as few weights as possible. At least how many weights must this be?

6. *Integer-dividing* an integer P by a natural number Q yields two other integers: the *remainder* of P on division by Q, which is at least 0 and at most $Q-1$, and the *quotient*, which is the greatest integer no bigger than the rational number P/Q. For instance, dividing 17 by 6 gives quotient 2 and remainder 5.
   Let n be any natural number. For each factor d of $n+1$, integer-divide n by d, and record the quotient and remainder. Prove that the sets of quotients and remainders you obtain by this process are the same.

Senior Division

1. For which prime numbers p and q (if any), is $5p^2q + 16pq^2$ a perfect square?

2. Let c_1, c_2, c_3, …, c_{2009} be a sequence of real numbers such that $|c_n - c_{n+1}| \leq 1$ for $1 \leq n \leq 2008$. Show that:
$$\left|\frac{c_1+c_2+\cdots+c_{2009}}{2009}-\frac{c_1+c_2+\cdots+c_{2008}}{2008}\right| \leq \frac{1}{2}.$$

3. Let ABC be a right-angled triangle with right angle at C. Pick a point D on the segment BC. Let E be a point on the circumcircle ω of ABD, such that DE is perpendicular to AB. Prove that $\angle BAE = \angle BEA$ if and only if AC is tangent to ω.

4. Let $n \geq 3$ be an odd integer. Determine the maximum possible value of the sum:

$$\sqrt{|x_1 - x_2|} + \sqrt{|x_2 - x_3|} + \cdots + \sqrt{|x_{n-1} - x_n|} + \sqrt{|x_n - x_1|}$$

where $0 \leq x_i \leq 1$ for $i = 1, 2, \ldots, n$.

5. Determine the minimum possible value of the expression $|n^2 - 5^{4m+3}|$ for non-negative integers m and n.

6. Michael's mother likes to keep him busy with an odd form of solitaire. To set up the game she places coins on some of the squares of a normal 8×8 chessboard. Michael plays by adding one coin at a time, always placing coins only on squares which already have at least two adjacent squares containing coins. (Two squares are adjacent if they share an edge, but not if they only share a vertex.)
Michael wins when he's placed a coin on every square of the board. What is the minimum number of coins that Michael's mother can place on the board to start with, so that it is still possible for Michael to win?