New Zealand Mathematical Olympiad Committee

NZMO Round One 2019 — Instructions

Submissions due date: 5th August 2019

The **New Zealand Mathematical Olympiad** (NZMO) consists of two rounds:

Round One (the NZMO1): A take home exam (the attached set of eight problems). Solutions are to be submitted by *5th August 2019*. Participation in the NZMO1 is open to any New Zealand intermediate or secondary student.

Round Two (the NZMO2): A three hour exam on *21st September 2019*. Participation in the NZMO2 is by invitation only. Invitations will be made based on the results of the NZMO1.

Awards for the NZMO (gold/silver/bronze/honourable mention) will be made based on the results of the NZMO2. Scores will not be announced for either round. Participants in the NZMO1 will only receive an indication of whether they have been invited to participate in the NZMO2; participants in the NZMO2 will just be told the level of their award, if any.

In addition, the results of both rounds of the NZMO will be used to select about 25 students to participate in the NZMOC training camp, to be held at the University of Auckland in January 2020. The training camp is the first step in the selection process to choose the team of six to represent New Zealand at the 2020 International Mathematical Olympiad, (IMO), to be held in Russia in July 2020. Only students who attend this camp are eligible for team selection.

General instructions:

- All solutions must be entirely your own work.
- You may not use a calculator, a computer or the internet (except as a reference, for e.g. definitions) to assist you in solving the problems.
- Although some problems may seem to require only a numerical answer, in order to receive full credit for the problem a complete justification must be provided. In fact, an answer alone will be worth at most 20% of the credit for a problem.
- We do not expect many, if any, perfect submissions. So, please submit all the solutions and partial solutions that you can find.
- Please address **all** queries about the problems to Chris Tuffley, c.tuffley@massey.ac.nz.

Any intermediate or secondary student can participate. Students submitting solutions must be New Zealand citizens or permanent residents who are currently enrolled in the New Zealand education system. If you are uncertain about your eligibility please contact Chris Tuffley (c.tuffley@massey.ac.nz) **before** submitting solutions.

Your solutions should be submitted online as a single PDF document at http://tiny.cc/nzmoc. Please complete the submission form **carefully**, especially your contact details. If you are having any difficulties with the submission form, contact Chris Tuffley (c.tuffley@massey.ac.nz).

July 2019
www.mathsolympiad.org.nz

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Notes

- There are eight problems. You should attempt to find solutions for as many as you can. Solutions (that is, answers with justifications) and not just answers are required for all problems, even if they are phrased in a way that only asks for an answer.
- Read and follow the "General instructions" accompanying these problems.
- If any clarification is required, please contact Chris Tuffley (c.tuffley@massey.ac.nz).

Problems

1. How many positive integers less than 2019 are divisible by either 18 or 21, but not both?
2. Find all real solutions to the equation
$$(x^2+3x+1)^{x^2-x-6}=1.$$
3. In triangle ABC, points D and E lie on the interior of segments AB and AC, respectively, such that $AD=1$, $DB=2$, $BC=4$, $CE=2$ and $EA=3$. Let DE intersect BC at F. Determine the length of CF.
4. Show that the number $122^n-102^n-21^n$ is always one less than a multiple of 2020, for any positive integer n.
5. Find all positive integers n such that $n^4-n^3+3n^2+5$ is a perfect square.
6. Let V be the set of vertices of a regular 21-gon. Given a non-empty subset U of V, let $m(U)$ be the number of distinct lengths that occur between two distinct vertices in U. What is the maximum value of $\frac{m(U)}{|U|}$ as U varies over all non-empty subsets of V?
7. Let $ABCDEF$ be a convex hexagon containing a point P in its interior such that $PABC$ and $PDEF$ are congruent rectangles with $PA=BC=PD=EF$ (and $AB=PC=DE=PF$). Let ℓ be the line through the midpoint of AF and the circumcentre of PCD. Prove that ℓ passes through P.
8. Suppose that $x_1,x_2,x_3,\dots x_n$ are real numbers between 0 and 1 with sum s. Prove that
$$\sum_{i=1}^{n}\frac{x_i}{s+1-x_i}+\prod_{i=1}^{n}(1-x_i)\le 1.$$