New Zealand Mathematical Olympiad Committee

Maths Workshop (Auckland East)

Friday July 5th, 6:00pm to 8:00pm

Macleans College, Kupe House

Problems

1. In how many ways can 33 be expressed as a sum of consecutive positive integers?
2. How many ways can a 7-letter word be constructed from the English alphabet if every letter must be distinct? (it doesn't matter if the word is not in the dictionary)
3. The Fibonnacci sequence $(1, 1, 2, 3, 5, 8, \ldots)$ is defined by $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 2$. Prove that $f_n^2 + 1$ is a multiple of both f_{n-1} and f_{n+1} for all even n.
4. A rhombus, ABCD, has sides of length 10. A circle with center A passes through C (the opposite vertex.) Likewise, a circle with center B passes through D. If the two circles are tangent to each other, what is the area of the rhombus?
5. Given 4 different squares such that the sum of their areas equals $\frac{1}{2}$, is it always possible to place them on a square board with area 1 without overlapping?
   What if there were 5 different squares such that the sum of their areas equals $\frac{1}{2}$?
6. An *lattice point* is any point (x, y), in the Cartesian plane such that both x and y are integers. Suppose A, B and C are lattice points such that there are no lattice points on sides: AB, BC or CA (apart from A, B and C themselves). Suppose also that the triangle ABC contains no lattice points in its interior. What is the greatest possible value of the area of triangle ABC?
7. A circle of radius r is inscribed into a triangle. Tangent lines to this circle parallel to the sides of the triangle cut out three smaller triangles. The radiuses of the circles inscribed in these smaller triangles are equal to 1, 2 and 3. Find r.
8. If x and y are positive real numbers, show that $x^y + y^x \geq 1$.