# New Zealand Mathematical Olympiad Committee

## Camp Selection Problems 2018 — Instructions

*Solutions due date: 28th September 2018*

These problems will be used by the NZMOC to select students for its International Mathematical Olympiad Training Camp, to be held in Auckland over 6–12 January 2019. Only students who attend this camp are eligible for selection to represent New Zealand at the 2019 International Mathematical Olympiad (IMO), to be held in Bath, the United Kingdom in July 2019. We have been fortunate in obtaining some sponsorship for the camp so have been able to keep the cost to $650, which covers all expenses for the camp including travel for those from outside Auckland.

At the camp a squad of 10–12 students will be chosen for further training. The New Zealand team for the 2019 IMO will be chosen from this squad. Those selected for the camp will also be invited to sit Round one of the British Mathematical Olympiad (BMO1), to be held on 1st December 2018.

Students will be selected into two groups for the camp: juniors and seniors.

- If you are currently in year 12, **or** you have been a member of the NZIMO training squad, then you will be considered only as a senior.
- Otherwise, you will be considered as either a junior or a senior.

Since the problems and the participants vary from year to year it is hard to be precise about the selection criteria. However, as a rough guide, if you solve five or more of the problems completely then you will be in the running for selection as a junior. Obviously, the criteria for senior selection are somewhat higher.

**General instructions:**

- All solutions must be entirely your own work.
- You may not use a calculator, a computer or the internet (except as a reference for e.g., definitions) to assist you in solving the problems.
- Although some problems may seem to require only a numerical answer, in order to receive full credit for the problem a complete justification must be provided. In fact, an answer alone will be worth at most 20% of the credit for a problem.
- We do not expect many, if any, perfect submissions. So, please submit all the solutions and partial solutions that you can find.
- Please address **all** queries about the problems to Chris Tuffley, c.tuffley@massey.ac.nz.

Students submitting solutions should be intending to remain in school in 2019 and should also hold New Zealand Passports or have New Zealand Resident status. To be eligible for the 2019 IMO you must have been born on or after 1st July 1999, and must not be formally enrolled in a University or similar institution prior to the IMO. If you are uncertain about your eligibility please contact Chris Tuffley (c.tuffley@massey.ac.nz) **before** submitting solutions.

Your solutions, together with a completed Registration Form (overleaf), should be sent to

> Dr Christopher Tuffley, Institute of Fundamental Sciences, Massey University Manawatū, Private Bag 11222, Palmerston North 4442, New Zealand

**arriving no later than 28th September 2018**. Please complete the registration form **carefully** and **legibly**, in particular your contact details. We regret that we are unable to accept electronic submissions. You will be notified whether or not you have been selected for the camp by 31st October 2018.

# Registration Form
# NZMOC Camp Selection Problems 2018

This form is a fillable PDF. You can type your answers into the form and then print and sign it to send in with your answers, or you can print it and then fill it in by hand.

**Name:** ______________________________

**Gender:** ______________ **School year level in 2019:** ________

**Home address:** ______________________________

______________________________

______________________________

**Email address:** ______________________________

**Home phone number:** ______________ **Cell:** ______________

**School:** ______________________________

**School postal address:** ______________________________

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______________________________

**Principal:** ______________________________

**HOD Mathematics:** ______________________________

| | | |
|---|---|---|
| Do you intend to take part in the camp selection problems for any other Olympiad camp? | yes | no |
| If so, and if selected, which camp would you prefer to attend? ______________ | | |
| Have you put your name forward for a Science camp or any other camp in January? | yes | no |
| Are there any criminal charges, or pending criminal charges against you? | yes | no |

Some conditions are attached to camp selection. You must be:

- Born on or after 1st July 1999
- Studying in 2019 at a recognised secondary school in NZ
- Available in July 2019 to represent NZ overseas as part of the NZIMO team if selected.
- A NZ citizen or hold NZ resident status.

**Declaration:** I satisfy these requirements, have worked on the questions without assistance from anyone else or the internet, and have read, understood and followed the instructions for the January camp selection problems. I agree to being contacted through the email address I have supplied.

Signature: ______________________________ Date: ______________

Attach this registration form to your solutions, and send them to

> Dr Christopher Tuffley, Institute of Fundamental Sciences, Massey University Manawatū, Private Bag 11222, Palmerston North 4442, New Zealand,

**arriving no later than 28th September 2018**.

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*Due: 28th September 2018*

## Notes

- If any clarification is required, please contact Chris Tuffley (c.tuffley@massey.ac.nz).
- There are ten problems and you should attempt to find solutions for as many as you can. Solutions (i.e., answers with justifications) and not just answers are required for all problems, even if they are phrased in a way that only asks for an answer.
- These are selection problems only: you will not receive a 'score', only an indication of whether you were selected.
- For further information, see the general instructions on the registration form that accompanies these problems.

## Problems

1. Suppose that $a$, $b$, $c$ and $d$ are four different integers. Explain why

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

must be a multiple of 12.

2. Find all pairs of integers $(a, b)$ such that

$$a^2 + ab - b = 2018.$$

3. Show that amongst any 8 points in the interior of a $7 \times 12$ rectangle, there exists a pair whose distance is less than 5.

*Note: The* interior *of a rectangle excludes points lying on the sides of the rectangle.*

4. Let $P$ be a point inside triangle $ABC$ such that $\angle CPA = 90^\circ$ and $\angle CBP = \angle CAP$. Prove that $\angle PXY = 90^\circ$, where $X$ and $Y$ are the midpoints of $AB$ and $AC$ respectively.

5. Let $a$, $b$ and $c$ be positive real numbers satisfying

$$\frac{1}{a+2019} + \frac{1}{b+2019} + \frac{1}{c+2019} = \frac{1}{2019}.$$

Prove that $abc \geq 4038^3$.

6. The intersection of a cube and a plane is a pentagon. Prove the length of at least one side of the pentagon differs from 1 metre by at least 20 centimetres.

7. Let $N$ be the number of ways to colour each cell in a $2 \times 50$ rectangle either red or blue such that each $2 \times 2$ block contains at least one blue cell. Show that $N$ is a multiple of $3^{25}$, but not a multiple of $3^{26}$.

8. Let $\lambda$ be a line and let $M, N$ be two points on $\lambda$. Circles $\alpha$ and $\beta$ centred at $A$ and $B$ respectively are both tangent to $\lambda$ at $M$, with $A$ and $B$ being on opposite sides of $\lambda$. Circles $\gamma$ and $\delta$ centred at $C$ and $D$ respectively are both tangent to $\lambda$ at $N$, with $C$ and $D$ being on opposite sides of $\lambda$. Moreover $A$ and $C$ are on the same side of $\lambda$. Prove that if there exists a circle tangent to all circles $\alpha, \beta, \gamma, \delta$ containing all of them in its interior, then the lines $AC$, $BD$ and $\lambda$ are either concurrent or parallel.

9. Let $x, y, p, n, k$ be positive integers such that

$$x^n + y^n = p^k.$$

Prove that if $n > 1$ is odd, and $p$ is an odd prime, then $n$ is a power of $p$.

10. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x)f(y) = f(xy + 1) + f(x - y) - 2$$

for all $x, y \in \mathbb{R}$.

*August 2018*

www.mathsolympiad.org.nz