New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2017 — Solutions

Due: 29th September 2017

1. *Alice has five real numbers $a < b < c < d < e$. She takes the sum of each pair of numbers and writes down the ten sums. The three smallest sums are 32, 36 and 37, while the two largest sums are 48 and 51. Determine e.*

   Solution: Out of the ten sums the largest is $d + e$, and the second largest is $c + e$. Therefore $d + e = 51$ and $c + e = 48$. Furthermore $a + b$ is the smallest sum and $a + c$ the second smallest, so $a + b = 32$ and $a + c = 36$.

   The third smallest sum could be either $a + d$ or $b + c$. However, we know that

$$a + d = (a + c) + (d + e) - (c + e) = 36 + 51 - 48 = 39.$$

   So $a + d$ cannot be the third smallest sum, and we must have $b + c = 37$. Combining what we've found so far we get

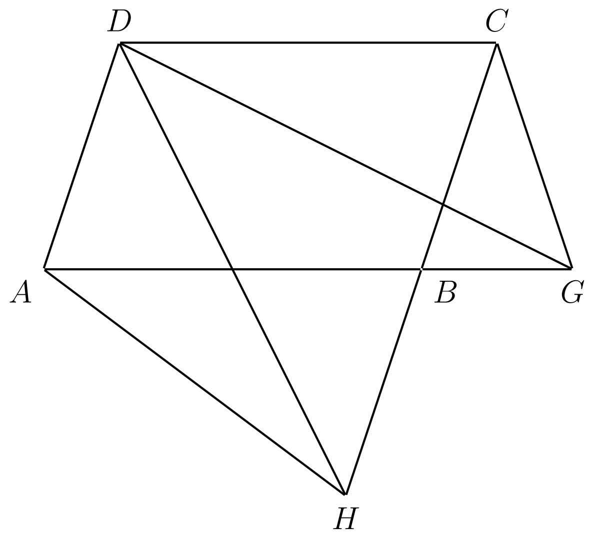
$$2e = 2(c + e) - (a + c) - (b + c) + (a + b) = 2 \cdot 48 - 36 - 37 + 32 = 55.$$

   Hence $e = 55/2$. □

2. *Let $ABCD$ be a parallelogram with an acute angle at A. Let G be the point on the line AB, distinct from B, such that $CG = CB$. Let H be the point on the line BC, distinct from B, such that $AB = AH$.*

   Prove that triangle DGH is isosceles.

   Solution: We show that triangles ADH and CGD are congruent, from which it follows that $DG = DH$.



   Using the fact that $ABCD$ is a paralleogram we have $CG = CB = DA$ and $AH = AB = DC$, so $AD = CG$ and $HA = DC$. Next observe that CBG and ABH are similar, because both are isosceles with apex at C and A, respectively, and $\angle CBG = \angle ABH$. Thus $\angle BAH = \angle BCG$. Then

$$\angle HAD = \angle HAB + \angle BAD = \angle BCG + \angle BCD = \angle GCD,$$

completing the proof that ADH and CGD are congruent (SAS). □

3. *Find all prime numbers p such that $16p + 1$ is a perfect cube.*

Solution: The equation $16p + 1 = n^3$ rearranges to

$$16p = n^3 - 1 = (n-1)(n^2 + n + 1).$$

Now n^2+n+1 is always odd, so 16 must divide $n-1$; and then p must divide n^2+n+1, because $n^2+n+1 > 1$. So $16 = n-1$, $p = n^2+n+1$, and we get $p = 17^2+17+1 = 307$, which is in fact prime. □

4. *Ross wants to play solitaire with his deck of n playing cards, but he's discovered that the deck is "boxed": some cards are face up, and others are face down. He wants to turn them all face down again, by repeatedly choosing a block of consecutive cards, removing the block from the deck, turning it over, and replacing it back in the deck at the same point. What is the smallest number of such steps Ross needs in order to guarantee that he can turn all the cards face down again, regardless of how they start out?*

Solution: $n/2$ for n even, and $(n+1)/2$ for n odd.

One possible strategy is as follows: If no more than half the cards are face up, turn them over one by one. This requires at most $n/2$ steps. If more than half the cards are face up, turn the entire deck over, and then turn the face up cards over one by one. For even n this takes at most $1 + (\frac{n}{2} - 1) = \frac{n}{2}$ steps, and for odd n it takes at most $1 + \frac{n-1}{2} = \frac{n+1}{2}$ steps.

To see that we can't do better, define the *degree of disagreement* to be the number of pairs of adjacent cards that are facing in opposite directions. For the purpose of calculating this we take two cards from a second deck and place one at the top of the deck and one at the bottom, both face down. These cards are not to be turned over, and account for whether the top and bottom cards are the correct way up. Then the degree of disagreement is 0 if and only if the entire deck is face down.

Observe that the degree of disagreement changes by at most 2 at each step, because it can change only at the boundaries of the block turned over. Suppose the deck starts with the cards alternately facing up and down, where for odd n we assume further that the top and bottom cards are face up. Then the degree of disagreement is n for n even and $n+1$ for n odd, so at least $n/2$ and $(n+1)/2$ steps are required, respectively.

Comments. Three other strategies that work to fix the deck in at most $n/2$ (n even) or $(n+1)/2$ (n odd) steps are the following:

(a) Divide the deck up into blocks of two adjacent cards, plus a single card by itself when n is odd: a total of $n/2$ or $(n+1)/2$ blocks, depending on whether n is even or odd. Each block can be fixed in at most one step, by turning both cards together if both are face up, or whichever is face up if just one is.

(b) At each step, turn the block of cards from the top-most face up card to the bottom-most face up card. The distance between the top-most and the bottom-most face up card deceases by at least two at each step, except at the last step when there is just one face up card.

(c) Divide the deck into blocks of face up and face down cards, where each block is as large as possible. The blocks alternate between face up and face down, so there can be at most $n/2$ or $(n+1)/2$ face up blocks, depending on whether n is even or odd. At each step, turn over one of the face up blocks.

The solution shows that the worst case is when the cards alternate face up and face down, starting with a face up card on top. However, to prove this really is the worst case we need to introduce the degree of disagreement (or something similar), so that we can measure how far the deck is from being fixed, and how much closer we can get at each step. $\square$

5. *Find all pairs (m, n) of positive integers such that the $m \times n$ grid contains exactly 225 rectangles whose side lengths are odd and whose edges lie on the lines of the grid.*

Solution: The possible pairs are $(1, 29)$, $(5, 9)$, $(9, 5)$ and $(29, 1)$.

The $m \times n$ grid has $m+1$ horizontal lines and $n+1$ vertical lines, and a "grid rectangle" corresponds to a choice of a pair of horizontal lines and a pair of vertical lines. Number the horizontal lines 1 to $m+1$, and the vertical lines 1 to $n+1$. A rectangle with odd side lengths is formed if and only if the pair of horizontal lines have opposite parity, and the pair of even lines have opposite parity.

Suppose first that at least one of m and n is odd. Without loss of generality we may assume that $m = 2k$. Then there are exactly $k+1$ odd-numbered and k even-numbered horizontal lines, for a total of $k(k+1)$ pairs of horizontal lines of opposite parity. This is even, and so the total number of rectangles with odd side lengths is even too, and in particular, cannot equal 225. We conclude that m and n must both be odd.

Let $m = 2k-1$, $n = 2\ell-1$. Then there are exactly k even- and k odd-numbered horizontal lines, and also exactly ℓ even- and ℓ odd-numbered vertical lines. In total then there are $k^2 \cdot \ell^2 = (k\ell)^2$ rectangles with odd sidelengths. From $(k\ell)^2 = 225$ we conclude that $k\ell = 15$. The possibilities are $(k, \ell) \in \{(1, 15), (3, 5), (5, 3), (15, 1)\}$, giving the solutions listed above. $\square$

6. *Let $ABCD$ be a quadrilateral. The circumcircle of the triangle ABC intersects the sides CD and DA in the points P and Q respectively, while the circumcircle of CDA intersects the sides AB and BC in the points R and S. The lines BP and BQ intersect the line RS in the points M and N respectively. Prove that the points M, N, P and Q lie on the same circle.*

Solution: By equality of angles subtended on the same chord, $\angle BAC = \angle BQC$ and $\angle CQP = \angle CBP$ (see Figure 1). In addition, quadrilateral $ACSR$ is cyclic, so $\angle RSC + \angle RAC = 180^\circ$, and

$$\begin{aligned} \angle BSR &= 180^\circ - \angle RSC && \text{(angles on a straight line)} \\ &= \angle RAC \\ &= \angle BAC \\ &= \angle BQC. \end{aligned}$$

Using these relations we obtain

$$\begin{aligned} 180^\circ - \angle PMN &= 180^\circ - \angle BMS && \text{(opposite angles)} \\ &= \angle SBM + \angle BSM && \text{(angles in triangle)} \\ &= \angle CBP + \angle BSR \end{aligned}$$

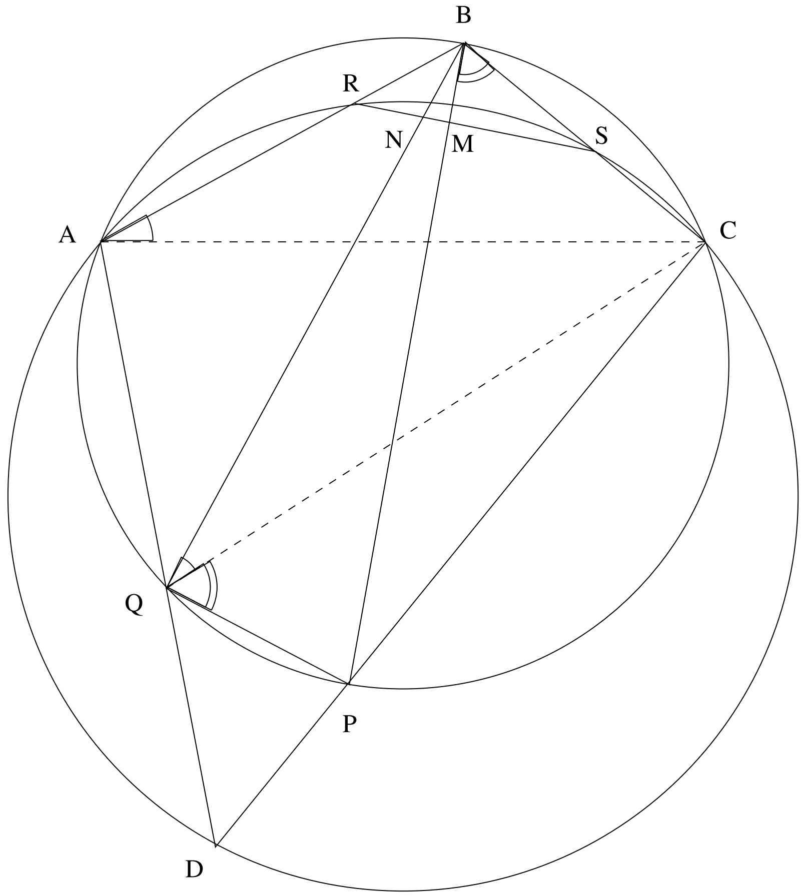


Figure 1: Diagram for Problem 6.

$$
\begin{aligned}
&= \angle CQP + \angle BQC \\
&= \angle BQP \\
&= \angle NQP,
\end{aligned}
$$

so $\angle PMN + \angle NQP = 180°$. This shows that $MNPQ$ is cyclic. $\square$

7. *Let a, b, c, d, e be distinct positive integers such that*

$$a^4 + b^4 = c^4 + d^4 = e^5.$$

Show that $ac + bd$ is composite.

Solution: Suppose to the contrary that $ac+bd = p$ is prime. Without loss of generality we may assume that $\max\{a, b, c, d\} = a$, and then because $a^4+b^4 = c^4+d^4$ we must have $\min\{a, b, c, d\} = b$. Note that $ac \equiv -bd \bmod p$, implying that $a^4c^4 \equiv b^4d^4 \bmod p$. Then mod p we have

$$b^4d^4 + b^4c^4 \equiv a^4c^4 + b^4c^4 = c^4(a^4 + b^4) = c^4(c^4 + d^4) = c^8 + c^4d^4,$$

from which it follows that $(c^4 + d^4)(c^4 - b^4) \equiv 0 \bmod p$. Thus, p must divide at least one of $c - b$, $c + b$, $c^2 + b^2$ and $c^4 + d^4$. Since b and c are distinct we have

$$0 < c - b < c + b < c^2 + b^2 < ac + bd = p,$$

so p must divide $c^4 + d^4 = e^5$. Thus $p^5 = (ac + bd)^5$ divides $c^4 + d^4$, but this is impossible because it's easily seen that $(ac + bd)^5 > c^4 + d^4$. $\square$

8. *Find all possible real values for a, b and c such that*

   (a) $a+b+c=51$*,*

   (b) $abc=4000$*,*

   (c) $0<a\le 10$ *and* $c\ge 25$*.*

Solution: Consider the polynomials

$$\begin{aligned} p(x) &= (x-a)(x-b)(x-c) = x^3-51x^2+kx-4000, \\ q(x) &= (x-10)(x-16)(x-25) = x^3-51x^2+810x-4000, \end{aligned}$$

where $k=ab+bc+ca$. In particular, the function $f(x)=p(x)-q(x)$ is a *linear function*; $f(x)=(k-810)x$ for any x. Now since $c\ge 25$, then we must have

$$(k-810)c = f(c) = p(c)-q(c) = 0-(c-10)(c-16)(c-25)\le 0.$$

Thus $(k-810)\le 0$. However, since $a\le 10$ we also get

$$(k-810)a = f(a) = p(a)-q(a) = 0-(a-10)(a-16)(a-25)\ge 0,$$

and since $a>0$, this implies $(k-810)\ge 0$. Combining these inequalites we conclude that $k-810=0$, so $k=810$.

It follows that the polynomials p and q are equal, and so have the same roots. Therefore a, b and c must equal 10, 16 and 25 in some order. Taking into account that $a\le 10$ and $c\ge 25$, we conclude that $a=10$, $b=16$ and $c=25$. □

9. *Let k and n be positive integers, with $k\le n$. A certain class has n students, and among any k of them there is always one that is friends with the other $k-1$. Find all values of k and n for which there must necessarily be a student who is friends with everyone else in the class.*

Solution: We construct a graph in which two students are joined by an edge precisely when they are *not* friends. The condition in the problem then states that among any k vertices, there is one that is not connected to any of the remaining $k-1$; more succinctly, the subgraph induced by any k vertices contains an isolated vertex. The problem is then to determine for which n and k the graph as a whole must have an isolated vertex. We will assume that the graph as a whole does not have an isolated vertex, and determine for which n and k we are able to satisfy the condition. The solution will then be those n and k for which we are never able to do this.

The assumption that the graph does not have an isolated vertex implies that every connected component has at least two vertices. Suppose first that every component has exactly two vertices (and so that n is necessarily even). If k is odd then any choice of k vertices will intersect at least one connected component in just one vertex, and the condition will be satisfied; but if $k=2m$ is even then we may choose k vertices that violate the condition by choosing m of the connected components. So the graph need not have an isolated vertex if n is even and k is odd.

Suppose now that there is at least one connected component C with at least 3 vertices. We claim that in this case it is possible to choose k vertices that violate the condition if $k\ge 2$. The vertices may be chosen as follows:

(a) Choose two adjacent vertices from C, and then two adjacent vertices from each of the other components in turn, stopping if at any point either k or $k-1$ vertices have been chosen.

(b) If fewer than k vertices were chosen at the previous step, then either $k-1$ vertices have been chosen, or two adjacent vertices have been chosen from each component. In the former case we obtain k vertices violating the condition by choosing a third vertex from C adjacent to one of those already chosen. Otherwise, we simply continue choosing vertices adjacent to those already chosen one by one until we have a total of k vertices. This is certainly possible, since there are $n \geq k$ vertices, and each vertex belongs to a component containing a vertex already chosen.

Since any graph satisfies the condition with $k = 1$, the above shows that the graph as a whole necessarily has an isolated vertex (that is, there is a student who is friends with everyone else) unless $k = 1$, or n is even and k is odd. □