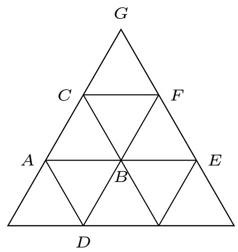
New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2016

Due: 28 September 2016

1. *Suppose that every point in the plane is coloured either black or white. Must there be an equilateral triangle such that all of its vertices are the same colour?*

   Solution: Yes there must be such a triangle. In fact consider any colouring and any 10 points in the plane arranged in a grid of equilateral triangles as below:

   Consider first the points ABC – obviously if all three are the same colour we've found a triangle. Otherwise two must be of the same colour, and third a different colour. By moving the triangular grid around a bit (including possibly a rotation) and possibly switching colours, we can assume that A and B are black and C is white. Now if D is black we have a black triangle ABD. If D is white, then if E is white we have a white triangle CDE, so E must be black. Now B and E are black, so if F is black we have a black triangle BEF, so F must be white. But now G is part of two triangles ABG where A and B are black and CFG where C and F are white – so no matter its colour we have a triangle with all its vertices of the same colour.

   Alternative: First we argue that there must be three equally spaced points on a line (like A, B and E) of the same colour. To see this, take a line, and take two points (say X and Y) on it of the same colour (say black). If their midpoint is black we've got the three points we want. Otherwise consider the two points W and Z such that X is the midpoint of WY and Y is the midpoint of XZ). Again, if either of these is black we're done, but if they're white we're done too (with the midpoint of XY we get three white points). Now referring to the original picture suppose ABE are all black. Then from equilateral triangles ABC, BEF, AEG all three of C, F and G must be white or we have a black equilateral triangle – but then we'd have a white one. □

2. *We consider 5×5 tables containing a real number in each of the 25 cells. The same number may occur in different cells, but no row or column contains five equal numbers. Such a table is* balanced *if the number in the middle cell of every row and column is the average of the numbers in that row or column. A cell is called* small *if the number in that cell is strictly smaller than the number in the cell in the very middle of the table. What is the least number of small cells that a balanced table can have?*

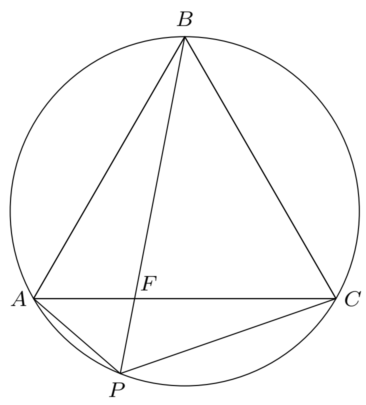
Solution: The answer is 3. The table below realises this.

4	4	3	4	0
4	4	3	4	0
3	3	0	3	−9
4	4	3	4	0
0	0	−9	0	−36

On the other hand consider the middle column and middle row – each must contain at least one small element since the average is equal to the central cell, and not all elements of a row or column can be equal. But now consider a small element in the middle column - it is the middle cell of its row so there must be an even smaller element in its row. Thus we certainly need at least one additional small element (giving three) and the example above shows that this suffices. □

3. *Consider an equilateral triangle ABC. Let P be an arbitrary point on the shorter arc AC of the circumcircle of ABC. Show that $PB = PA + PC$.*

Solution: Consider the following diagram:



Note that $\angle APB = \angle CPB = 60°$ since each subtends a chord of the equilateral triangle (and angles subtended by a common chord of a circle are equal). So triangle APB is similar to triangle FAB, so $PA/PB = FA/AB$. By the same argument on the other side of PB, $PC/PB = FC/BC$. So:

$$\frac{PA}{PB} + \frac{PC}{PB} = \frac{FA}{AB} + \frac{FC}{BC}.$$

But $BC = AB = FA + AC$ so the right hand side is 1 which implies $PA + PC = PB$.

Alternative: Ptolemy's theorem for cyclic quadrilaterals says that

$$BP \times AC = AB \times PC + BC \times AP.$$

But $AC = AB = BC$, so $AC = PC + AP$. □

4. *A quadruple (p, a, b, c) of positive integers is a* karaka quadruple *if*

- *p is an odd prime number*
- *a, b and c are distinct, and*
- *$ab+1$, $bc+1$ and $ca+1$ are divisible by p.*

(a) *Prove that for every karaka quadruple (p, a, b, c) we have*

$$p + 2 \leq \frac{a+b+c}{3}.$$

(b) *Determine all numbers p for which a karaka quadruple (p, a, b, c) exists with*

$$p + 2 = \frac{a+b+c}{3}.$$

Solution: We use the notation $x|y$ to mean that y is divisible by x. Suppose without loss of generality that $a < b < c$. None of a, b or c can be divisible by p. On the other hand

$$\begin{aligned} &p|(bc+1) - (ab+1) \\ &p|bc - ab \\ &p|b(c-a) \end{aligned}$$

Since b is not a multiple of p and p is prime, $p|c-a$. Similarly $p|b-a$. So $b = a + jp$ and $c = a + kp$ for some positive integers $j < k$. Therefore

$$\frac{a+b+c}{3} = a + \left(\frac{j+k}{3}\right) p \geq a + p$$

so we will have the result we want unless $a = 1$, $b = p+1$, $c = 2p+1$. But then $ab + 1 = p + 2$ and since p is an odd prime, that's not divisible by p.

For the second half notice that the same argument shows that for equality to hold we must have $a = 2$, $b = p+2$, $c = 2p+2$. Then $ab+1 = 2p+5$ and for this to be a multiple of p we need $p = 5$. It's easy to check that $p = 5$ satisfies the given conditions as well.

□

5. *Find all polynomials $P(x)$ with real coefficients such that the polynomial*

$$Q(x) = (x+1)P(x-1) - (x-1)P(x).$$

is constant.

Solution: Obviously any constant polynomial $P(x) = c$ has this property. So suppose that we have a polynomial $P(x)$ of degree $n \geq 1$ with the property. Let $P(x) = ax^n + bx^{n-1} + R(x)$ where $a \neq 0$ and $R(x)$ is a polynomial of degree less than $n-1$. Consider the coefficient of x^n in $Q(x)$. Since the term $R(x)$ cannot produce such a coefficient, it is the same as the coefficient of x^n in:

$$(x+1)(a(x-1)^n + b(x-1)^{n-1} - (x-1)(ax^n + bx^{n-1}).$$

This is easily computed to be:

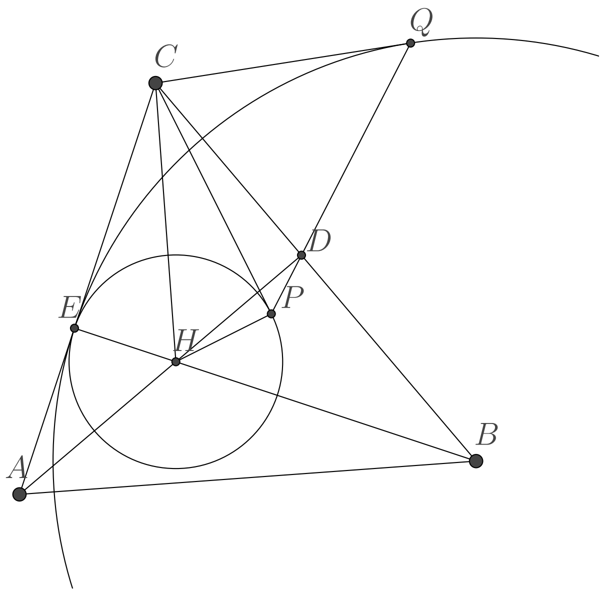
$$(-na + a + b) - (-a + b) = (-n + 2)a.$$

So, for $Q(x)$ to be constant in this case we must have $n = 2$, i.e., $P(x)$ is a quadratic. Finally considering $P(x) = ax^2 + bx + c$ we can check that $Q(x)$ is constant only if $a = b$.

That is, the polynomials $P(x)$ with the given property are all those of the form ax^2+ax+c for any real numbers a (including 0) and c. □

6. *Altitudes AD and BE of an acute triangle ABC intersect at H. Let $P \neq E$ be the point of tangency of the circle with radius HE centred at H with its tangent line going through point C, and let $Q \neq E$ be the point of tangency of the circle with radius BE centred at B with its tangent line going through C. Prove that the points D, P and Q are collinear.*

Solution: If $AC = BC$ then $D = P$ and there is nothing to prove so assume throughout that $AC \neq BC$. There are two possible configurations depending on whether $AC < BC$ or $AC > BC$. The diagram for the first one is given below, and the argument refers to that configuration. If $AC > BC$ then P lies between D and Q but essentially the same argument still applies (modulo some sign changes).



Since $\angle HPC = \angle HDC = 90°$, $CHPD$ all lie on the circle with diameter CH. Therefore $\angle HPD = 180° - \angle HCD$. Also

$$\begin{aligned}\angle CPD &= \angle HPD - \angle HPC \\ &= (180° - \angle HCD) - 90° \\ &= 90° - \angle HCD.\end{aligned}$$

Now $PC = CQ$ so $\angle CPQ = 90° - \angle PCQ/2$. But $\angle PCQ = \angle ECQ - \angle ECP = 2\angle ECD - 2\angle ECH$. So

$$\angle CPQ = 90° - (\angle ECD - \angle ECH) = 90° - \angle HCD.$$

Since $\angle CPD = \angle CPQ$, the points P, D and Q are collinear. $\square$

7. *Find all positive integers n for which the equation*

$$(x^2 + y^2)^n = (xy)^{2016}$$

has positive integer solutions.

Solution: Suppose that we have a solution (x, y) for some value of n. Let $d = \gcd(x, y)$ and say $x = ad$, $y = bd$. Since $xy \leq x^2 + y^2$, $n \leq 2016$ Then:

$$(a^2 + b^2)^n = (ab)^{2016} d^{4032-2n}$$

So a and b are both divisors of the left hand side, but are relatively prime to the left hand side – this is only possible if $a = b = 1$. Thus we have:

$$2^n = d^{4032-2n}.$$

Conversely, if we have a solution to this equation, then $x = y = d$ is a solution to the original. So $d = 2^k$ for some k and

$$n = k(4032 - 2n)$$
$$n = \frac{4032k}{2k + 1}$$

So $2k + 1$ is an odd divisor of $4032 = 64 \times 63$ (since k and $2k + 1$ are relatively prime). Going through the cases gives n must be one of 1344, 1728, 1792, 1920, or 1984. $\square$

8. *Two positive integers r and k are given as is an infinite sequence of positive integers $a_1 \leq a_2 \leq a_3 \leq \cdots$ such that $\frac{r}{a_r} = k + 1$. Prove that there is a positive integer t such that $\frac{t}{a_t} = k$.*

Solution: Proof by Contradiction. Assume that $a_t \neq t/k$ for all t. *i.e.* $a_k \neq 1$ and $a_{2k} \neq 2$ and $a_{3k} \neq 3$ and so on. However, since (a_n) is an increasing sequence of positive integers, so this means

$$a_k \geq 2 \quad \text{and} \quad a_{2k} \geq 3 \quad \text{and} \quad a_{3k} \geq 4 \quad \text{and so on.}$$

Now let $a_r = x$ (which is an integer). Therefore

$$a_{xk} \geq x + 1 = a_r + 1.$$

Since (a_n) is an increasing sequence, this implies that $xk > r$. However this contradicts the fact that $x = \frac{r}{k+1}$. $\square$

9. *An n-tuple $(a_1, a_2 \ldots, a_n)$ is* occasionally periodic *if there exist a non-negative integer i and a positive integer p satisfying $i + 2p \leq n$ and $a_{i+j} = a_{i+j+p}$ for every $j = 1, 2, \ldots, p$. Let k be a positive integer. Find the least positive integer n for which there exists an n-tuple $(a_1, a_2 \ldots, a_n)$ with elements from the set $\{1, 2, \ldots, k\}$, which is not occasionally periodic but whose arbitrary extension $(a_1, a_2, \ldots, a_n, a_{n+1})$ is occasionally periodic for any $a_{n+1} \in \{1, 2, \ldots, k\}$.*

Solution: We claim that the shortest such sequence has length $n = 2^k - 1$. Note that a sequence is occasionally periodic if it has a repeated block (of at least one character) in it somewhere, i.e., is of the form X, B, B, Y where X, B and Y are themselves sequences. If a sequence (using the elements 1 through k) is not occasionally periodic, but extending it by one character always is then the repeated block must always occur at the end of the extended sequence.

We can construct sequences of length $2^k - 1$ inductively, start with $A_1 = 1$ and then given a sequence A_{k-1} using the numbers 1 through $k-1$, extend it to $A_k = A_{k-1}, k, A_{k-1}$. So:

$$
\begin{aligned}
A_1 &= 1 \\
A_2 &= 2, 1, 2 \\
A_3 &= 2, 1, 2, 3, 2, 1, 2 \\
A_4 &= 2, 1, 2, 3, 2, 1, 2, 4, 2, 1, 2, 3, 2, 1, 2 \\
&\dots
\end{aligned}
$$

Extending by k gives repeated blocks A_{k-1}, k while extending by a smaller element gives a repeated block at the end of A_{k-1}. There can be no other repeated blocks since k only occurs once and (inductively) there are no blocks within A_{k-1}.

To prove that no longer sequences of this type are possible we consider a weaker condition - namely we just consider sequences whose one character extensions have a repeated block at the end. We claim that the shortest such sequences (and hence certainly the shortest sequences of the type we are interested in) have length $2^k - 1$. Proceeding by induction again this is obviously true for $k = 1$. Now suppose we have a shortest such sequence, S, on the characters 1 through k. For different extensions i and j the blocks at the end must differ, i.e., if $S, i = X, B_i, i, B_i, i$ and $S, j = Y, B_j, j, B_j, j$ then for $i \neq j$, $B_i \neq B_j$ (if they were equal the preceding character in the original sequence would be both i and j). Since the names of the characters don't matter, let's suppose that extending by k gives the longest possible repeated block: $S = X, B, k, B$. Because the other blocks are all shorter than B, no character in X occurs in any repeated block, so because S is shortest, X must be empty, i.e., $S = B, k, B$. Furthermore, let B' be the result of deleting all the k characters from B. Then it is easy to see that B' extended by any character from 1 through $k-1$ ends in a repeated block (just extend B by the same character, find the repeated block we know exists there, and delete all the k's from it.) So the length of B' (and hence the length of B) is at least $2^{k-1} - 1$ by the inductive hypothesis, and the length of S is at least $2 \times (2^{k-1} - 1) + 1 = 2^k - 1$. □