New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2016

Due: 28 September 2016

1. Suppose that every point in the plane is coloured either black or white. Must there be an equilateral triangle such that all of its vertices are the same colour?

2. We consider 5×5 tables containing a real number in each of the 25 cells. The same number may occur in different cells, but no row or column contains five equal numbers. Such a table is *balanced* if the number in the middle cell of every row and column is the average of the numbers in that row or column. A cell is called *small* if the number in that cell is strictly smaller than the number in the cell in the very middle of the table. What is the least number of small cells that a balanced table can have?

3. Consider an equilateral triangle ABC. Let P be an arbitrary point on the shorter arc AC of the circumcircle of ABC. Show that $PB = PA + PC$.

4. A quadruple (p, a, b, c) of positive integers is a *karaka quadruple* if
   - p is an odd prime number
   - a, b and c are distinct, and
   - $ab + 1$, $bc + 1$ and $ca + 1$ are divisible by p.

   (a) Prove that for every karaka quadruple (p, a, b, c) we have
   $$p + 2 \leq \frac{a + b + c}{3}.$$

   (b) Determine all numbers p for which a karaka quadruple (p, a, b, c) exists with
   $$p + 2 = \frac{a + b + c}{3}.$$

5. Find all polynomials $P(x)$ with real coefficients such that the polynomial
   $$Q(x) = (x + 1)P(x - 1) - (x - 1)P(x).$$
   is constant.

6. Altitudes AD and BE of an acute triangle ABC intersect at H. Let $P \neq E$ be the point of tangency of the circle with radius HE centred at H with its tangent line going through point C, and let $Q \neq E$ be the point of tangency of the circle with radius BE centred at B with its tangent line going through C. Prove that the points D, P and Q are collinear.

7. Find all positive integers n for which the equation
   $$(x^2 + y^2)^n = (xy)^{2016}$$
   has positive integer solutions.

8. Two positive integers r and k are given as is an infinite sequence of positive integers $a_1 \leq a_2 \leq a_3 \leq \cdots$ such that $\frac{r}{a_r} = k + 1$. Prove that there is a positive integer t such that $\frac{t}{a_t} = k$.

9. An n-tuple $(a_1, a_2 \ldots, a_n)$ is *occasionally periodic* if there exist a non-negative integer i and a positive integer p satisfying $i + 2p \leq n$ and $a_{i+j} = a_{i+j+p}$ for every $j = 1, 2, \ldots, p$. Let k be a positive integer. Find the least positive integer n for which there exists an n-tuple $(a_1, a_2 \ldots, a_n)$ with elements from the set $\{1, 2, \ldots, k\}$, which is not occasionally periodic but whose arbitrary extension $(a_1, a_2, \ldots, a_n, a_{n+1})$ is occasionally periodic for any $a_{n+1} \in \{1, 2, \ldots, k\}$.