New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2015

Due: 23th September

1. *Starting from the number* 1 *we write down a sequence of numbers where the next number in the sequence is obtained from the previous one either by doubling it, or by rearranging its digits (not allowing the first digit of the rearranged number to be* 0*). For instance we might begin:*

$$1, 2, 4, 8, 16, 61, 122, 212, 424, \ldots$$

Is it possible to construct such a sequence that ends with the number 1000000000*? Is it possible to construct one that ends with the number* 9876543210*?*

Solution:

- The number $10^9 = 1000000000$ can be reached in the following steps:
  (a) iteratively double 1 to reach 512,
  (b) rearrange to get 125,
  (c) iteratively double until 512×10^3,
  (d) rearrange to get 125×10^3,
  (e) iteratively double until 512×10^6,
  (f) rearrange to get 125×10^6,
  (g) iteratively double until 10^9.
- First note that 9876543210 is a multiple of 3. Note that any rearrangement of the digits will never change the number's residue modulo 3. Moreover, if a number n is not a multiple of 3, then neither is $3n$. Therefore no multiple of 3 can ever be reached. Hence 9876543210 can never occur in such a sequence.

□

2. *A mathematics competition had* 9 *easy, and* 6 *difficult problems. Each of the participants in the competition solved* 14 *of the* 15 *problems. For each pair of problems, consisting of an easy and a difficult problem, the number of participants who solved both those problems was recorded. The sum of these recorded numbers was* 459*. How many participants were there?*

Solution: Let the number of contestants be $x + y$, where: x of the contestants solved all the easy problems, and y of the contestants solved all the hard problems. Let a pair consisting of an easy problem and a difficult problem be called a *counting-pair*. There are $6 \times 9 = 54$ counting-pairs in total. So x contestants solved $9 \times 5 = 45$ counting pairs, and y contestants solved $8 \times 6 = 48$ counting-pairs. Thus the total is

$$45x + 48y = 459.$$

Since $x \geq 0$, this implies that $y \leq 9$. Now taking this equation mod 45 yields $3y \equiv 9 \pmod{45}$ and thus $y \equiv 3 \pmod{15}$. Therefore $y = 3$. And thus $x = 7$. Therefore there were $x + y = 10$ participants in total. □

3. *Let ABC be an acute angled triangle. The arc between A and B of the circumcircle of ABC is reflected through the line AB, and the arc between A and C of the circumcircle of ABC is reflected over the line AC. Obviously these two reflected arcs intersect at the point A. Prove that they also intersect at another point inside the triangle ABC.*

   Solution: Let the other point of intersection of these two arcs be H. Since ABC is acute, both $\angle AHB$ and $\angle AHC$ must be obtuse. Thus $\angle BHC$ is less than $180°$ and hence H lies inside triangle ABC. □

4. *For which positive integers m does the equation:*

$$(ab)^{2015} = (a^2 + b^2)^m$$

   have positive integer solutions?

   Solution: First, by AM-GM, we have $a^2 + b^2 \geq 2ab > ab$. Therefore if $m \geq 2015$ then we would have

$$(ab)^{2015} = (a^2 + b^2)^m \geq (a^2 + b^2)^{2015} > (ab)^{2015},$$

   which is a contradiction. Therefore $m < 2015$. Let $g = \gcd(a, b)$ and let $a = gx$ and $b = gy$, so x and y are relatively prime. The equation now becomes

$$g^{2(2015-m)}(xy)^{2015} = (x^2 + y^2)^m.$$

   If x were greater than 1, then there would be some prime $p \mid x$. Since $\gcd(x, y) = 1$ we must have $p \not| y$. However this is a contradiction since p would be a divisor of $g^{2(2015-m)}(xy)^{2015}$ but not a divisor of $(x^2 + y^2)^m$. Hence $x = 1$. Similarly $y = 1$ and thus:

$$g^{2(2015-m)} = 2^m.$$

   Therefore g is a power of 2. Let $g = 2^n$ so $2n(2015 - m) = m$. This equation rearranges to give:

$$(2n + 1)(2015 - m) = 2015.$$

   Hence $(2015 - m)$ must be a divisor of 2015. Since $2015 = 5 \times 13 \times 31$, the divisors of 2015 are $1, 5, 13, 31, 65, 155, 403, 2015$ and thus:

$$m = 0, 1612, 1860, 1950, 1984, 2002, 2010 \text{ or } 2014.$$

□

5. *Let n be a positive integer greater than or equal to 6, and suppose that $a_1, a_2, a_3, \ldots, a_n$ are real numbers such that the sums $a_i + a_j$ for $1 \le i < j \le n$, taken in some order, form consecutive terms of an arithmetic progression $A, A+d, A+2d, \ldots, A+(k-1)d$, where $k = \frac{n(n-1)}{2}$. What are the possible values of d?*

   Solution: Wlog assume $a_1 \le a_2 \le a_3 \le \cdots \le a_n$. So the smallest term in the arithmetic progression must be equal to $a_1 + a_2$. The second smallest term must be$a_1 + a_3$ and therefore the difference must be

   $$|d| = a_3 - a_2.$$

   Similarly, the largest term is $a_n + a_{n-1}$ and the second largest term is $a_n + a_{n-2}$ and so

   $$|d| = a_{n-1} - a_{n-2}.$$

   However, this implies $a_2 + a_{n-1} = a_3 + a_{n-2}$. Since $n \ge 4$, this implies that two *different* terms in the arithmetic progression are equal, and therefore $d = 0$. □

6. *In many computer languages, the division operation ignores remainders. Let's denote this operation by //, so for instance $13//3 = 4$. If, for some b, $a//b = c$ then we say that c is a* near factor *of a. Thus, the near factors of 13 are 1, 2, 3, 4 and 6. Let a be a positive integer. Prove that every positive integer less than or equal to $\sqrt{a}$ is a near factor of a.*

   Solution: Suppose c is any integer with $1 \le c \le \sqrt{a}$. Let $b = \lfloor \frac{a}{c} \rfloor$. Thus $b < \frac{a}{c}$ which implies $cb \le a$, and hence $\frac{a}{b} \ge c$. Moreover, since $c \le \sqrt{a}$, we get $b \ge \lfloor \sqrt{a} \rfloor$ and since both b and c are integers, this means $b \ge c$. So

   $$\frac{a}{c} < b + 1$$
   $$a < bc + c < bc + b$$
   $$\frac{a}{b} < c + 1.$$

   Hence c is an integer this implies $bc \le a + c - 1$.

   □

7. *Let ABC be an acute-angles scalene triangle. Let P be a point on the extension of AB past B, and Q a point on the extension of AC past C such that $BPQC$ is a cyclic quadrilateral. Let N be the foot of the perpendicular from A to BC. If $NP = NQ$ then prove that N is also the circumcentre of APQ.*

   Solution: First, since $BPQC$ is cyclic, we get $\angle APQ = \angle ACB$ and $\angle AQP = \angle ABC$, so triangle APQ is scalene and thus A does not lie on the perpendicular bisector of PQ. Now let O be the actual circumcentre of APQ, and let M be the midpoint of PQ. Now since $OB = OC$, we know that O lies on the perpendicular bisector of BC. Similarly since $NP = NQ$, we know that N also lies on the perpendicular bisector of BC. Hence O lies on the line MN.
   Now since $\angle ANB = 90°$, and O is the circumcentre of APQ, we can get

   $$\angle OAB = \angle OAP = 90° - \frac{1}{2}\angle AOQ = 90° - \angle APQ = 90° - \angle ABN = \angle NAB.$$

Therefore O lies on the lines AN. However, since APQ is scalene, lines MN and AN can only intersect at one point. Hence $O = N$. □

8. *Determine all positive integers n which have a divisor d with the property that $dn + 1$ is a divisor of $d^2 + n^2$.*

Solution: Let $n = dx$ so $(dn + 1) = d^2x + 1$ and $(d^2 + n^2) = d^2(x^2 + 1)$. Note also that $\gcd(d, dn + 1) = 1$. Since $d^2x + 1 \mid d^2(x^2 + 1)$, we must have $d^2x + 1 \mid x^2 + 1$. Hence $d^2x + 1 \leq x^2 + 1$ and thus $d^2 \leq x$. Note also that

$$d^4(x^2 + 1) - (d^2x + 1)(d^2x - 1) = d^4 + 1$$

is a multiple of $d^2x + 1$. Hence $d^2x + 1 \leq d^4 + 1$ and thus $x \leq d^2$. Since

$$x \leq d^2 \leq x$$

we must have $x = d^2$ and so $n = d^3$. Now we check that all cube numbers work; if $n = d^3$ then $dn + 1$ will always be a divisor of $d^2 + n^2$ because

$$d^2 + n^2 = d^2(1 + d^4) = d^2(dn + 1).$$

□

9. *Consolidated Megacorp is planning to send a salesperson to Elbonia who needs to visit every town there. It is possible to travel between any two towns of Elbonia directly either by barge or by mule cart (the same type of travel is available in either direction, and these are the only types of travel available). Show that it is possible to choose a starting town so that the salesperson can complete a round trip visiting each town exactly once and returning to her starting point, while changing the type of transportation used at most one time (this is desirable, since it?s hard to arrange for the merchandise to be transferred from barge to cart or vice versa).*

Solution: Let each town be a vertex in a complete graph. For any two vertices, if the type travel between them is barge, colour the edge red, and if it is mule, colour the edge blue. Now let k be the length of the longest path in this graph consisting of at most one 'colour-change'. Wlog, let

$$u_1, u_2, u_3, \ldots, u_a, v_1, v_2, \ldots, v_b$$

be such a path (*i.e.* $k = a + b$) where u_iu_{i+1} are red for all i, and v_jv_{j+1} are blue for all j, and u_av_1 is blue. Assume for the sake of contradiction, that there exists a vertex x not in this path. Consider the colour of edge xu_a.

- If xu_a is red, then we can increase the length of this path by inserting x between u_a and v_1.
- If xu_a is blue, then we can increase the length of this path by inserting x between u_{a-1} and u_a.

Both cases contradict the maximality of this path - so no such vertex x can exist. Therefore the salesperson can achieve this task by starting at vertex u_a and then completing the path in the following manner:

$$u_a, v_1, v_2, \ldots, v_b u_1, u_2, u_3, \ldots, u_a.$$

□