New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2015

Due: 23 September 2015

Notes:

- If any clarification is required, please contact Michael Albert (michael.albert@cs.otago.ac.nz).
- There are nine problems and you should attempt to find solutions for as many as you can. Solutions (i.e., answers with justifications) and not just answers are required for all problems, even if they are phrased in a way that only asks for an answer.
- These are selection problems only – you will not receive a 'score', only an indication of whether or not you were selected.
- For further information, see the general instructions on the registration form that accompanies these problems.

Questions:

1. Starting from the number 1 we write down a sequence of numbers where the next number in the sequence is obtained from the previous one either by doubling it, or by rearranging its digits (not allowing the first digit of the rearranged number to be 0). For instance we might begin:
$$1, 2, 4, 8, 16, 61, 122, 212, 424, \ldots$$
Is it possible to construct such a sequence that ends with the number 1,000,000,000? Is it possible to construct one that ends with the number 9,876,543,210?

2. A mathematics competition had 9 easy, and 6 difficult problems. Each of the participants in the competition solved 14 of the 15 problems. For each pair, consisting of an easy and a difficult problem, the number of participants who solved both those problems was recorded. The sum of these recorded numbers was 459. How many participants were there?

3. Let ABC be an acute angled triangle. The arc between A and B of the circumcircle of ABC is reflected through the line AB, and the arc between A and C of the circumcircle of ABC is reflected over the line AC. Obviously these two reflected arcs intersect at the point A. Prove that they also intersect at another point inside the triangle ABC.

4. For which positive integers m does the equation:
$$(ab)^{2015} = (a^2 + b^2)^m$$
have positive integer solutions?

5. Let n be a positive integer greater than or equal to 6, and suppose that a_1, a_2, $\ldots$,a_n are real numbers such that the sums $a_i + a_j$ for $1 \leq i < j \leq n$, taken in some order, form consecutive terms of an arithmetic progression A, $A + d$, $\ldots A + (k-1)d$, where $k = n(n-1)/2$. What are the possible values of d?

6. In many computer languages, the division operation ignores remainders. Let's denote this operation by $//$, so for instance $13//3 = 4$. If, for some b, $a//b = c$, then we say that c is a *near factor* of a. Thus, the near factors of 13 are 1, 2, 3, 4, and 6. Let a be a positive integer. Prove that every positive integer less than or equal to $\sqrt{a}$ is a near factor of a.

7. Let ABC be an acute-angled scalene triangle. Let P be a point on the extension of AB past B, and Q a point on the extension of AC past C such that $BPQC$ is a cyclic quadrilateral. Let N be the foot of the perpendicular from A to BC. If $NP = NQ$ then prove that N is also the centre of the circumcircle of APQ.

8. Determine all positive integers n which have a divisor d with the property that $dn + 1$ is a divisor of $d^2 + n^2$.

9. Consolidated Megacorp is planning to send a salesperson to Elbonia who needs to visit every town there. It is possible to travel between any two towns of Elbonia directly either by barge or by mule cart (the same type of travel is available in either direction, and these are the only types of travel available). Show that it is possible to choose a starting town so that the salesperson can complete a round trip visiting each town exactly once and returning to her starting point, while changing the type of transportation used at most one time (this is desirable, since it's hard to arrange for the merchandise to be transferred from barge to cart or vice versa).