New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2014

Due: 24 September 2014

1. *Prove that for all positive real numbers a and b:*

$$\frac{(a+b)^3}{4} \geq a^2b + ab^2.$$

Solution: The simplest approach is simply to clear fractions and note that the right hand side factors. So an equivalent inequality is:

$$(a+b)^3 \geq 4ab(a+b).$$

Since $a+b$ is positive, dividing by $a+b$ leaves an equivalent inequality

$$(a+b)^2 \geq 4ab$$

which is either directly equivalent to the arithmetic mean - geometric mean inequality or reduces to $(a-b)^2 \geq 0$ by expansion and simplification. $\square$

2. *Let ABC be a triangle in which the length of side AB is 4 units, and that of BC is 2 units. Let D be the point on AB at distance 3 units from A. Prove that the line perpendicular to AB through D, the angle bisector of $\angle ABC$, and the perpendicular bisector of BC all meet at a single point.*

Solution: Let E be the midpoint of AB. The first line is the perpendicular bisector of BE. The triangle BEC is isosceles so the second line is also the perpendicular bisector of EC. Thus the three lines are the perpendicular bisectors of the sides of BEC and so meet at a single point. $\square$

3. *Find all pairs (x, y) of positive integers such that $(x+y)(x^2+9y)$ is the cube of a prime number.*

Solution: Suppose that $(x+y)(x^2+9y) = p^3$. Since $x+y < x^2+9y$ it must be the case that $x+y = p$ and $x^2+9y = p^2$. Therefore:

$$\begin{aligned}(x+y)^2 &= x^2+9y \\ x^2+2xy+y^2 &= x^2+9y \\ (2x+y)y &= 9y \\ 2x+y &= 9.\end{aligned}$$

So we must have $x+y$ prime and $2x+y = 9$. It is easily checked that the only possible pairs are $(2, 5)$ and $(4, 1)$. $\square$

4. *Given 2014 points in the plane, no three of which are collinear, what is the minimum number of line segments that can be drawn connecting pairs of points in such a way that adding a single additional line segment of the same sort will always produce a triangle of three connected points?*

   Solution: Once some line segments have been chosen, call two points connected if it is possible to go from one to the other via existing line segments. Clearly, if there are two points that are not connected then we can add the line segment between them without creating a triangle. When there are no segments, there are 2014 different groups of connected points (each point is alone in its group). Adding a single line segment reduces the number of groups by at most 1. So, as long as there are fewer than 2013 line segments there will be at least two groups and another one can be added without creating a triangle. On the other hand, if we choose a single point, and the segments connecting it to all the others, then we have 2013 line segments, and any additional ones will create a triangle. So the answer is 2013. □

5. *Let ABC be an acute angled triangle. Let the altitude from C to AB meet AB at C' and have midpoint M, and let the altitude from B to AC meet AC at B' and have midpoint N. Let P be the point of intersection of AM and BB' and Q the point of intersection of AN and CC'. Prove that the point M, N, P and Q lie on a circle.*

   Solution: Triangles $BB'A$ and $CC'A$ are similar (since they both have a right angle and share $\angle BAC$). Since AN is the median of $BB'A$ and AM is the corresponding median of $CC'A$, $\angle AMC' = \angle ANB'$ or in other words $\angle PMQ = \angle PNQ$. This condition is well-known to imply that $PMNQ$ are concyclic. □

6. *Determine all triples of positive integers a, b and c such that their least common multiple is equal to their sum.*

   Solution: Suppose without loss of generality that $a \leq b \leq c$. The three numbers cannot be equal since then their least common multiple would be the same. So c is a divisor of $a + b + c$ which is less than $3c$. Hence $a + b + c = 2c$, and so $a + b = c$. We can divide a, b and c by their greatest common factor without affecting the property, so we may assume that their greatest common factor is 1. In particular, c must be odd (if it were even, then either both a and b would be even, contradicting the least common factor property), or both would be odd, but then since each was a divisor of $2c$ each would be a divisor of c and the least common multiple would be c not $2c$. So, c is an odd number which can be expressed as a sum of two numbers - one a factor of itself, and the other twice a factor of itself. But if p were a prime dividing c and either one of the two factors, it would also divide the other (contradicting the greatest common factor condition). So the only possibility is that the two numbers are 1 and 2, with $c = 3$. In other words the solutions to the original problem are all the triples $(k, 2k, 3k)$ for any positive integer k and in any order. □

7. *Determine all pairs of real numbers (k, d) such that the system of equations:*

$$x^3 + y^3 = 2$$
$$kx + d = y,$$

has no solutions (x, y) with x and y real numbers.

Solution: Substituting from the second equation in the first and rearranging gives:

$$(k^3 + 1)x^3 + 3dk^2x^2 + 3d^2kx + d^3 - 2 = 0.$$

If $k^3 \neq -1$ this is a cubic with a non-zero lead coefficient and so has at least one real solution x (and hence $y = kx + d$ is also real). So, in order for there to be no solutions we must have $k = -1$ and the quadratic:

$$3dx^2 - 3d^2x + d^3 - 2 = 0$$

can have no real solutions. If $d = 0$ this is clearly the case. If $d \neq 0$ then there will be no real solutions exactly if the discriminant of the quadratic is negative. This discrimant is:

$$(-3d^2)^2 - 4 \cdot 3d \cdot (d^3 - 2) = 3d(3d^3 - 4d^3 + 8) = 3d(8 - d^3),$$

which is negative for $d < 0$ and $d > 2$.

Thus the system has no real solutions exactly for the pairs $(-1, d)$ where $d \leq 0$ or $d > 2$. □

8. *Michael wants to arrange a doubles tennis tournament among his friends. However, he has some peculiar conditions: the total number of matches should equal the total number of players, and every pair of friends should play as either teammates or opponents in at least one match. The number of players in a single match is four. What is the largest number of people who can take part in such a tournament?*

   Solution: Each match creates six pairs among the four players who take part. So, in n matches, at most $6n$ pairs are created. The total number of pairs is $n(n-1)/2$. So to fulfil the conditions it is necessary that $n(n-1)/2 \leq 6n$, i.e. $(n-1)/2 \leq 6$, or $n \leq 13$. So certainly there cannot be more than 13 players.

   To see that it is possible to include 13 players, consider them as numbered from 0 through 12. Now in match i (also for $0 \leq i \leq 12$) let players i, $i+2$, $i+3$ and $i+7$ take part (wrapping around modulo 13). Consider the set of differences among these four numbers: $\{2, 3, 7, 1, 5, 4\}$. Note that these are the same modulo 13 as $\{1, 2, 3, 4, 5, 6\}$. Since any two players have a difference of at most 6, all pairs will play in at least one (in fact exactly one) match. □

9. *Let AB be a line segment with midpoint I. A circle, centred at I has diameter less than the length of the segment. A triangle ABC is tangent to the circle on sides AC and BC. On AC a point X is given, and on BC a point Y is given such that XY is also tangent to the circle (in particular X lies between the point of tangency of the circle with AC and C, and similarly Y lies between the point of tangency of the circle with BC and C. Prove that $AX \cdot BY = AI \cdot BI$.*

   Solution: Let $\angle YXA = 2x$ and $\angle XYB = 2y$. From quadrilateral $ABXY$, $\angle XAB + \angle YBA = 360° - 2x - 2y$. But these two angles are equal so $\angle XAB = \angle YBA = 180° - x - y$. Also $\angle AXI = x$ since XA and XY are tangent to the circle, and I is its centre. So $\angle XIA = y$. Similarly, $\angle YIB = x$. So triangles AXI and BIY are similar. Hence $AX/AI = BI/BY$ or $AX \cdot BY = AI \cdot BI$. □

10. *In the land of Microbablia the alphabet has only two letters, 'A' and 'B'. Not surprisingly, the inhabitants are obsessed with the band ABBA. Words in the local dialect with a high ABBA-factor are considered particularly lucky. To compute the ABBA-factor of a word you just count the number of occurrences of ABBA within the word (not necessarily consecutively). So for instance AABA has ABBA-factor 0, ABBA has ABBA-factor 1, AABBBA has ABBA-factor 6, and ABBABBA has ABBA factor 8. What is the greatest possible ABBA-factor for a 100 letter word?*

Solution: We first show that in words of a given length, the greatest ABBA-factor occurs in a word of the form A ... A B ... B A ... A.To see this, consider any word that has an A occurring between two B's. Count the number, l, of A's to its left, and the number r of A's to its right. Suppose that $l \geq r$. Move the A under consideration to the very right hand end of the word. This does not affect any ABBA occurrences not involving this letter. Any ABBA occurrence involving this letter and letters to its left still exists, but we have lost the ABBA occurrences involving this letter as the initial A, and one of the r A's to its right (as the final A). But we have gained at least this many occurrences of ABBA since we can pair these occurrences with new ones using one of the l A's originally to its left, and the letter we moved. So the new word has at least as many occurrences of ABBA as the old one. If $l < r$ we can do a similar move but putting the A at the very left hand end. After a series of such moves we will have a word whose ABBA-factor is at least as great as the original, and which is of the form we want.

Now consider words of this form - by a similar argument if the difference between the lengths of the initial block of A's and the final block of A's is two or more we can increase the ABBA-factor by moving an A from the longer block to the shorter block (there must be at least two B's since otherwise the ABBA-factor is 0 and we know we can do better than that!)

So we need only to consider words of the form (x A's)(y B's)(x A's) and (x A's)(y B's)($x-1$ A's). For words of the first type the ABBA-factor is $x^2y(y-1)/2$ and for words of the second type it is $x(x-1)y(y-1)/2$. Consider the effect of changing a word of the second type to the first type by changing a B to A. The difference in the ABBA-factor between the new word and the old one is one half of:

$$x^2(y-1)(y-2) - x(x-1)y(y-1) = x(y-1)(x(y-2) - (x-1)y) = x(y-1)(y-2x).$$

So the ABBA factor increases or is unchanged if $y \geq 2x$ and decreases otherwise. Similarly a change from the first type to the second (still changing a B to an A) decreases the ABBA-factor if $y < 2x+2$ and leaves it unchanged or increases it otherwise. Summarising, we can increase the ABBA-factor by changing a B to an A exactly if the number of B's in the original word is more than twice as great as twice the length of the larger of the two blocks of A's in the new word. Thereafter we always decrease the ABBA-factor.

In particular for words of length 100 the greatest possible ABBA-factor will occur with (25 A's)(50 B's)(25 A's) and is equal to $25^3 \times 49$. □